Vector Functions and Curves
Applications

Question

Solve the initial value problem

$$\begin{aligned}\frac{d\underline{r}}{dt} &= \underline{k}\times\underline{r}\\ \underline{r}(0) &= \underline{i}+\underline{k}\end{aligned}$$

Describe the curve $\underline{r}=\underline{r}(t)$.

Answer

$$\begin{aligned}\frac{d\underline{r}}{dt} &= \underline{k}\times\underline{r}\\ \underline{r}(0) &= \underline{i}+\underline{k}\end{aligned}$$

If $\underline{r}(t)=x(t)\underline{i}+y(t)\underline{j}+z(t)\underline{k}$
Then

$$\begin{aligned}x(0) &= z(0)=1\\ y(0) &= 0\end{aligned}$$

As $\underline{k}\bullet(d\underline{r}/dt)=\underline{k}\bullet(\underline{k}\times\underline{r})=0$, velocity is always perpendicular to $\underline{k}$, meaning that $z(t)$ must be constant.
$z(t)=z(0)=1$, $\forall t$
$\Rightarrow$

$$\frac{dx}{dt}\underline{i}+\frac{dy}{dt}\underline{j}=\frac{d\underline{r}}{dt}=\underline{k}\times\underline{r}=xunj=y\underline{i}$$

In component form this becomes

$$\begin{aligned}\frac{dx}{dt} &= -y\\ \frac{dy}{dt} &= x\\ \Rightarrow\frac{d^2x}{dt^2} &= -\frac{dy}{dt}=-x\\ x &= A\cos t+B\sin t\end{aligned}$$

As $x(0)=1$ and $y(0)=0$, $\Rightarrow A=1$ and $B=0$.

$$\Rightarrow x(t)=\cos t$$

$$y(t) = \sin t$$

So the path has equation

$$\underline{r}=\cos t\underline{i}+\sin t\underline{j}+\underline{k}$$