Question

Show that $C(x) = \frac{1}{4}(x+2)$ defines a topological conjugacy between $f(x) = 4x(1-x)$ and $g(x) = 2 - x^2$. Deduce that g has chaotic orbits.

Answer

$$\left.\begin{array}{rcl} f \circ C(x) & = & 4 \cdot \frac{1}{4}(x+2)\left(1 - \frac{1}{4}(x+2)\right) = \frac{1}{4}(x+2)(2-x) \\ C \circ g(x) & = & \frac{1}{4}92 - x^2 + 2) = \frac{1}{4}(4 - x^2) = \frac{1}{4}(2+x)(2-x) \end{array}\right\}$$

so $f \circ C = C \circ g$.

Since f has orbits which are bounded and not asymptotically periodic the same holds for g. For positive Lyapunov exponents we use the fact that $C'(x) = \frac{1}{4}$ so from $g(x) = C^{-1}(f(C(x)))$ we get $g'(x) = 4f'(C(x)) = f'(C(x))$, so the Lyapunov exponent of the g-orbit of x is the same as the Lyapunov exponent of the f-orbit of $C(x)$.

[We could not expect this for a general conjugacy C if e.g. $C'(x) \to 0$ somewhere.]