Question

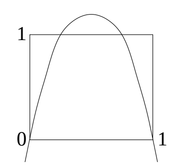
What are the options for the Lyapunov exponents of bounded orbits of $f_a(x) = ax(1-x)$ whene (i)a=2.5, (b)a=3.1?
Show that if $a > 2 + \sqrt{5}$ then every orbit has positive Lyapunov exponent (if it exists).

Answer

All orbits of f which remain bounded are attracted to the fixed point $x_* = 1 - \dfrac{1}{2.5} = 0.6$, except for 0 which remains at 0, and 1 with $f(1) = 0$. $f'(0.6) = 2.5(1-1.2) = -0.5$ and $f'(0) - 2.5$ so the only options for Lyapunov exponents are $\underline{\ln(2.5)}$ for (0,1) or $\underline{\ln(0.5)} < 0$ for all other points of [0,1].

For g there are repelling fixed points at 0 and $x_* = 1 - \dfrac{1}{3.1} = 0.68$ approx, with Lyapunov exponents
$\underline{\ln|g'(0)| = \ln(3.1) > 0}$ and $\underline{\ln|g'(0.68)| = \ln(1.1) > 0}$, and an attracting 2-cycle $\{p, q\}$ with $p,\ q$ roots of $x^2 - 1.32x + 0.43$.
Then $(g^2)'(p) = g'(q)g'(p) = (3.1)^2(1-2q)(1-2p) = (3.1)^2(1-2(p+q)+4pq) = (3.1)^2(1-2.54+1.72) = 0.77$, so the Lyapunov exponent for p and hence for all points which lie in (0,1) but do not land on x_* is $\underline{\dfrac{1}{2}\ln(0.77) = \ln(0.88) \text{ approx}, < 0.}$

For h(x) = ax(1 − x) with a > 4 the graph looks like:



so the Lyapunov exponent of any bounded orbit will be > 0 <u>if</u> the slope of the graph is everywhere > 1 or < -1.

Now the graph cuts the top of the square where $x = \dfrac{1}{2}\left(1 \pm \sqrt{1 - \dfrac{4}{a}}\right)$, where the slope is $h'(x) = \mp\sqrt{a^2 - 4a}$. This gives $|h'(x)| > 1$ when $a^2 - 4a > 1$: $a > 2 + \sqrt{5}$.