Question

(i) Show that if $f'(x) = (x-a)(x-b)(x-c)$ with $a,\ b,\ c$ all distinct then the Schwarzian derivative $Sf(x)$ is negative everywhere (where defined).

(ii) Show that if $Sf(x) < 0$ and $S_g(x) < 0$ for all x then $S(f \circ g)(x) < 0$.

Answer

(i) $\dfrac{f"(x)}{f'(x)} = \dfrac{1}{(x-a)} + \dfrac{1}{(x-b)} + \dfrac{1}{(x-c)}$

$$\frac{f'''(x)}{f'(x)} = \frac{2}{(x-b)(x-c)} + \frac{2}{(x-c)(x-a)} + \frac{2}{(x-a)(x-b)}$$

so $\dfrac{f'''(x)}{f'(x)} - \left(\dfrac{f"(x)}{f'(x)}\right)^2 = -\left(\dfrac{1}{(x-a)^2} + \dfrac{1}{(x-b)^2} + \dfrac{1}{(x-c)^2}\right) < 0$

and so certainly $Sf(x) < 0$.

(ii)

$$\begin{aligned} \text{Write } h &= f(g) \text{ to denote } h(x) = f(g(x0): \text{ then} \\ h' &= f'(g)(g')^2 + f'(g)g'' \\ h'' &= f''(g)(g')^2 + f'(g)g'' \\ h''' &= f'''(g)(g')^3 + 3f''(g)g'g'' + f'(g)g''' \end{aligned}$$

and we find

$2h'h''' - 3(h'')^2 = [2f'(g)f'''(g) - 3(f''(g))^2](g')^4 + 2(f'(g))^2[2g'g''' - 3(g'')^2]$

which gives the result since both square brackets [ ] are < 0.