Question
Let $f_a 9x) = a - x^2$.
Find:

(i) the value a_1 of a such that f_a has exactly one fixed point,

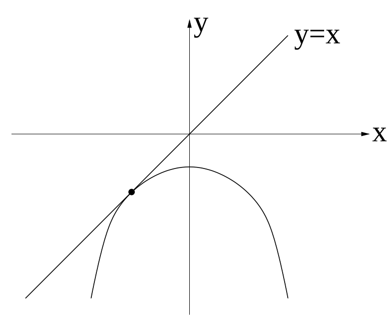
(ii) the largest value a_2 of a for which $f - a$ has <u>no</u> 2-cycle,

(iii) the value a_3 of a at which an attracting 2-cycle becomes repelling.

Show that the conditions of the period-doubling theorem are satisfied for f_a^2 and also $f_{a_3}^2$ (at the appropriate points).

Answer

(i) Exactly one fixed point when the parabola $y - ax^2$ is tangent to the line $y = x$, i.e. equation $a - x^2 = x$ has repeated roots. Condition "$b^2 - 4ac$" here is $1 = -4a$, i.e. $\underline{a_1 = -\frac{1}{4}}$



(ii) $f_a^2(x) = a-(a-x^2)^2$, so fixed points of f_a^2 where $x = a-(a-x^2)^2$, that is $x^4-2ax^2+x-(a-a^2 0 = 0$. Left hand side vanishes at fixed points of f_a, so has (x_2+x-a) as a factor: we find LHS= $(x^2+x-a)(x^2-x+(1-a))$. Thus per-2 points are roots of $x^2 - x + (1 - a) = 0$; these are real if and only if $a > \frac{3}{4}$ = max a with no 2-cycle.

(iii) If $\{p, q\}$ is a 2-cycle then $(f_a^2)'(p) = f_a'(q)f_a'(p) = 4pq$ since ${f_a}'(x) = -2x$. Thus 2-cycle repelling when $4q = -1$, i.e. $pq = -\frac{1}{4}$. But $pq =$ product of roots of $(x^2 - x + (1 - a)) = 0$, i.e. $pq = (1 - a)$. Therefore 2-cycle becomes repelling where $-\frac{1}{4} = (1 - a)$, i.e. $\underline{a_3 = \frac{5}{4}}$.

We have $f_a'(x) = -2x$. When $a = a_2 = \dfrac{3}{4}$ the fixed points are $x = \dfrac{1}{2},\ -\dfrac{3}{2}$ so $\underline{f'_{a_2}\left(\dfrac{1}{2}\right) = -1.}$

Now $(f_a^2)(x) = f_a'(f_a(x)) \cdot f_a'(x) = -2(a - x^2) \cdot -2x$ giving $(f_a^2)'\left(\dfrac{1}{2}\right) = 2a - \dfrac{1}{2}$; then $\underline{\dfrac{\partial}{\partial a}(f_a^2)'\left(\dfrac{1}{2}\right)}\Big|_{a=a_2} = 2 \neq 0$.

[Since $2 > 0$ and $Sf_a < 0$ the bifurcation is supercritical.]

For the 2-cycle $\{p, q\}$ we have $\underline{(f_{a_3}^2)'(p) = -1}$ (that's how a_3 was found). Now $(f_a^4)'(x) = 16 f_a^3(x) \cdot f_a^2(x) \cdot f_a(x) \cdot x$ (Chain Rule). We have $\dfrac{\partial}{\partial a} f_a(x) = 1$, $\dfrac{\partial}{\partial a} f_a^2(x) = 1 - 2f_a(x)$, $\dfrac{\partial}{\partial a} f_a^3(x) = 1 - 2f_a^2(x)(1 - 2f_a(x))$ (using $f_a^2(x) = a - (f_a(x))^2$, $f_a^3(x) = a - (f_a^2(x))^2$) and if $\{p, q\}$ is a 2-cycle for f_a these give $\underline{\dfrac{\partial}{\partial a} f_a(p) = 1}$, $\underline{\dfrac{\partial}{\partial a} f_a^2(p) = 1 - 2q}$, $\underline{\dfrac{\partial}{\partial a} f_a^3(p) = 1 - 2p + 4pq}$. We use these to differentiate $(f_a^4)'(p)$ as a product:

$\dfrac{\partial}{\partial a}(f_a^4)'(p) = 16[(1 - 2p + 4pq)pq + q(1 - 2q)q + qp1]p$. When $a = a_3 = \dfrac{3}{4}$, p and q are the roots of $x^2 - x - \dfrac{1}{4} = 0$ so $pq = -\dfrac{1}{4}$, $p + q = 1$. So $\underline{\dfrac{\partial}{\partial a}(f_a^4)'(p)} = 16\left[\dfrac{1}{2}p^2 - \dfrac{1}{4}q(1 - 2q) - \dfrac{1}{4}p\right] = 8(p^2 + q^2) - 4 = 8(p+q)^2 \underline{= 8 \neq 0.}$

[Since $8 > 0$ and $Sf_a^2 < 0$ (since $Sf_a < 0$) the bifurcation from a 2-cycle to a 4-cycle at $a = a_3$ is supercritical.]