QUESTION

(a) State the Duality Theorem of linear programming, and use it to prove the Theorem of Complementary Slackness.

(b) Use duality theory to determine whether $x_1 = 3$, $x_2 = 0$, $x_3 = -2$, $x_4 = 0$, is an optimal solution of the linear programming problem

$$\begin{array}{ll} \text{maximize} & z = 10x_1 - 8x_2 + 8x_3 + 15x_4 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0 \\ & 7x_1 - 2x_2 + 3x_3 + 7x_4 \leq 15 \\ & 8x_1 + 5x_2 + 4x_3 - 2x_4 \leq 18 \\ & 4x_1 + 3x_2 - 2x_3 - 3x_4 = 16. \end{array}$$

Analyze whether your conclusion alters if the objective is changed to

$$z' = 10x_1 - 5x_2 + 8x_3 + 17x_4$$

but the constraints are unaltered.

(c) If a linear programming problem has a unique optimal solution, then is the dual guaranteed to have a unique optimal solution? Justify your answer by providing either a proof or a counter example.

ANSWER

(a) The duality theorem states that:

- if the primal problem has an optimal solution, then so has the dual, and $z_p = z_D$;
- if the primal problem is unbounded, then the dual is infeasible;
- if the primal problem is infeasible, then the dual is either infeasible or unbounded.

Consider the following primal and dual problems

$$\begin{array}{llll} \text{Maximize} & z_p = \mathbf{c}^T\mathbf{x} & \text{Minimize} & z_D = \mathbf{b}^T\mathbf{y} \\ \text{subject to} & \mathbf{x} \geq), \mathbf{s} \geq \mathbf{0} & \text{subject to} & \mathbf{y} \geq), \mathbf{t} \geq \mathbf{0} \\ & A\mathbf{x} + \mathbf{s} = \mathbf{b} & & A^T\mathbf{y} - \mathbf{t} = \mathbf{c} \end{array}$$

For optimal solution, complementary slackness states that $y_i s_i = 0$ ($i = 1, \ldots, m$), $t_j x_j = 0$ ($j = 1, \ldots n$).

For feasible solutions of the primal and dual, we have

$$z_p = \mathbf{c}^T\mathbf{x} = (\mathbf{y}^T A - \mathbf{t}^T)\mathbf{x} = \mathbf{y}^T(\mathbf{b} - \mathbf{s}) - \mathbf{t}^T\mathbf{x} = z_D - \mathbf{y}^T\mathbf{s} - \mathbf{t}^T\mathbf{x}$$

For an optimal solution of the primal and the dual, $z_D = z_D$ so

$$\mathbf{y}^T\mathbf{s} + \mathbf{t}^T\mathbf{x} = 0$$

Since variables are non-negative this implies that

$$y_i s_i = 0,\ i = 1, \dots, m$$

$$t_j x_j = 0,\ j = 1, \dots, n$$

(b) For the given solution $z = 14,\ s_1 = 0,\ s_2 = 2$, where s_1, s_2 are the two slack variables.

The dual problem is

$$\begin{array}{ll} \text{minimize} & z_D = 15y_1 + 18y_2 + 16y_3 \\ \text{subject to} & y_1 \geq 0,\ y_2 \geq 0 \\ & 7y_1 + 8y_2 + 4y_3 \geq 10 \\ & -2y_1 + 5y_2 + 3y_3 \geq -8 \\ & 3y_1 + 4y_2 - 2y_3 = 8 \\ & 7y_1 - 2y_2 - 3y_3 = 15 \end{array}$$

Complementary slackness gives $y_2 = 0,\ t_1 = 0$ where t_1, t_2 are the slack variables for the dual. Therefore

$$\begin{aligned} 7y_1 + 4y_3 &= 10 \\ 3y_1 - 2y_3 &= 8 \\ 7y_1 - 3y_3 &= 15 \end{aligned}$$

The first two equations yield $y_1 = 2$ and $y_3 = -1$, which does not satisfy the third equation. Therefore the given solution is not optimal.

For the modified problem, the first, third and fourth constraints are satisfied for the new right hand sides of 10, 8,17 respectively. However, the left hand side of constraint 2 is -7, which is less than the new right hand side of -5, so the solution is still non optimal.

(c) An obvious counter-example is when the primal is degenerate. For example

$$\begin{array}{ll} \text{maximize} & z = 2x_1 + 3x_2 \\ \text{subject to} & x_1 \geq 0, x_2 \geq 0 \\ & x_1 + 2x_2 \leq 2 \\ & 2x_1 + x_2 \geq 3 \end{array}$$

has $x_1 = 2,\ x_2 = 0,\ z = 4$ as a unique optimal solution, but its dual

$$\begin{array}{ll} \text{minimize} & z_D = 2y_1 + 4y_2 \\ \text{subject to} & y_1 \geq 0, y_2 \geq 0 \\ & y_1 + 2y_2 \geq 2 \\ & 2y_1 + y_2 \geq 3 \end{array}$$

Has $y_1 = 2,\ y_2 = 0,\ z = 4$ and $y_1 = \frac{4}{3},\ y_2 = \frac{1}{3},\ z = 4$ as alternative optimal solutions.