QUESTION

(a) Explain two approaches by which linear programming is used to tackle multi-objective linear programming problems.

(b) Describe three alternative pivoting rules in linear programming, and state the situations for which they are most appropriate.

(c) From given observations $(x_1, y_1), \ldots, (x_n, y_n)$, it is required to find values of a and b to be used in the linear model $y = ax + b$. The values of a and b are to be chosen so that

$$\sum_{i=1}^{n} |y_i - ax_i - b|$$

is minimized. Give a linear programming formulation of this problem.

(d) Solve the following linear programming problem using the dual simplex method.

$$\begin{array}{ll} \text{Minimize} & z = 15x_1 + 6x_2 + 22x_3 \\ \text{subject to} & x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0 \\ & 5x_1 + 2x_2 + 7x_3 \geq 7 \\ & 6x_1 + 3x_2 + 5x_3 \geq 9. \end{array}$$

Find all other optimal solutions.

ANSWER

(a) If $z_1, \ldots, z_r$ are the objective functions to be maximised, one approach is to minimize the composite objective

$$\alpha_1 z_1 + \ldots + \alpha_r z_r$$

for suitable weights $\alpha_1, \ldots, \alpha_r$.

Another approach is to evaluate the trade-offs by maximizeingz_1 subject to $z_2 \geq k_2, \ldots z_r \geq k_r$ for varying values of the constants $k_2, \ldots, k_r$.

(b)

- The standard pivoting rule is to select a pivot column with the smallest (negative) objective coefficient.
- Bland's smallest subscript rule chooses the pivot column so that the entering variable has the smallest subscript amoung candidates with a negative objective coefficient. If there is a choice (equal ratios) the leaving variable is chosen so that it has the smallest subscript. This rule is used to avoid cycling in degenerate problems.

- The largest increase pivoting rule chooses a pivot column so that the corresponding pivot element gives the largest increase in the objective function. This helps to overcome badly scaled variables.

(c)

Minimize $z = u_1 + v_1 + \ldots + u_n + v_n$

subject to

$$u_0 \geq 0, \ldots u_n \geq 0$$
$$v_1 \geq 0, \ldots, v_n \geq 0$$
$$ax_1 + b + u_1 - v_i = y_1$$
$$\vdots$$
$$ax_n + b + u_n - v_n = y_n$$

(d) Introduce slack variables $s_1 \geq 0,\ s_2 \geq 0$, and maximize $z' = z$

Basic	z'	x_1	x_2	x_3	s_1	s_2	
s_1	0	-5	-2	-7	1	0	-7
s_2	0	-6	-3	-5	0	1	-9
	1	15	6	22	0	0	0
Ratio		$\frac{5}{2}$	2	$\frac{22}{5}$			

Basic	z'	x_1	x_2	x_3	s_1	s_2	
s_1	0	-1	0	$-\frac{11}{3}$	1	$-\frac{2}{3}$	-1
x_2	0	2	1	$\frac{5}{3}$*0	$-\frac{1}{3}$	3	
	1	3	0	12	0	2	-18
Ratio		3		$\frac{36}{5}$		3	

Basic	z'	x_1	x_2	x_3	s_1	s_2	
x_1	0	1	0	$\frac{11}{3}$	-1	$\frac{2}{3}$	1
x_2	0	0	1	$-\frac{17}{3}$	2	$-\frac{5}{3}$	1
	1	0	0	1	3	0	-21

Optimal solution is $x_1 = 1,\ x_2 = 1,\ x_3 = 0,\ z = 21$. To obtain an alternative optimal solution, perform an ordinary simplex iteration.

Basic	z'	x_1	x_2	x_3	s_1	s_2	
s_2	0	$\frac{3}{2}$	0	$\frac{11}{2}$	$-\frac{3}{2}$	1	$\frac{3}{2}$
x_2	0	$\frac{5}{2}$	1	$\frac{7}{2}$	$-\frac{1}{2}$	0	$\frac{7}{2}$
	1	0	0	1	3	0	-21

An alternative optimal solution is $x_1 = 0,\ x_2 = \frac{7}{2},\ x_3 = 0,\ z = 21$

Any optimal solution is of the form

$$(x_1, x_2, x_3) = \lambda(1, 1, 0) + (1 - \lambda)(0, \frac{7}{2}, 0)$$

for $0 \leq \lambda \leq 1$.