QUESTION

(a) Let $(G, *)$ be a group. Carefully prove the following statement using only the axioms for a group, indicating which axiom you used at each stage of the argument: There is a unique element $h \in G$ such that $h * g = g$ for every element $g \in G$.

(b) State Lagrange's Theorem and use it to prove that if p and q are prime and G is a group of order pq then every proper subgroup G is cyclic. (A proper subgroup is one not equal to G.)

(c) Write out the Cayley table for the group of symmetries of an equilateral triangle using the notation s to represent the anticlockwise rotation through $\pi/3$ and x, y, z to denote the three reflections in the lines X, Y, Z as marked in figure 1.

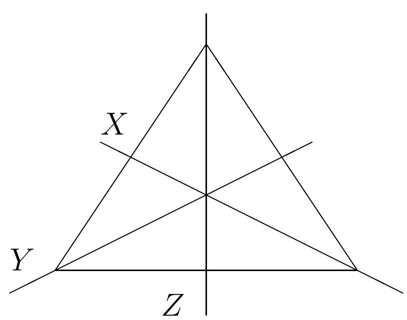

Figure 1

(d) For each of the statements below either show it is true OR give an example to show that it is false:

(i) If g and h are elements of a group and both have order n then their product also has order n.

(ii) If every proper subgroup of a group G is cyclic then so is G.

(iii) Every group of order 8 contains a cyclic subgroup of order 4.

ANSWER

(a) By the identity axiom there is an element $e \in G$ such that $e * g = g \forall g \in G$. Now suppose $h \in G$ and $h * g = g \forall g \in G$. In particular $h * h = h$.

By the inverse axiom there is an element $h^{-1} \in G$ such that $h^{-1} * h = e$ so $h^{-1} * (h * h) = h^{-1} * h = e$.

By associativity $(h^{-1} * h) * h = h^{-1} * (h * h)$ so $e * h = (h^{-1} * h) * h = h^{-1} * (h * h) = e$ or $h = e$.

(b) Lagrange's Theorem

If G is a finite group and H is a subgroup of G then $|H|$ divides $|G|$.

If $|G| = pq$ with p, q prime then any proper subgroup $h < G$ has $|H| = 1, p$ or q.

Since p, q are prime H is cyclic.

(c)

$\circ$	e	s	s^2	x	y	z
e	e	s	s^2	x	y	z
s	s	s^2	e	y	z	x
s^2	s^2	e	s	z	x	y
x	x	z	y	e	s^2	s
y	y	x	z	s	e	s^2
z	z	y	x	s^2	s	e

OR

$\circ$	e	s	s^2	x	y	z
e	e	s	s^2	x	y	z
s	s	s^2	e	z	x	y
s^2	s^2	e	s	y	z	x
x	x	z	y	e	s	s^2
y	y	z	x	s^2	e	s
z	z	x	y	s	s^2	e

(d) **(i)** False, $x, y \in D_3$ above have order 2, $xy = s^2$ has order 3.

(ii) False, Every proper subgroup of D_3 is cyclic but D_3 is not.

(iii) False $C_2 \times C_2 \times C_2$