QUESTION

(a) For real numbers a and bn let $a * b$ denote the number $a + b + 2ab$.

(i) Show that the set $X = \{x \in R | x \neq -\frac{1}{2}\}$ is closed under the binary operation $*$.

(ii) Say what it means for a binary operation to be associative and show that the binary operation $*$ is associative.

(iii) Show that there is a unique real number r such that $r*s = s = s*r$ for every real number s, and that this lies in X.

(iv) Show that every element of X has an inverse under the binary operation $*$.

(b) For each positive integer n less than or equal to 8 give a list of groups of order n so that no two groups on your list in your list are isomorphic, but every group of order n is isomorphic to one of the groups on your list. For each order where there is more than one isomorphism class of group of that order, indicate how you distinguish the different isomorphism classes.

ANSWER

(a) **(i)** For any $a, b \in X,\ a + b + 2ab \in R$.

Suppose $a+b+2ab = -\frac{1}{2}$. Then $a+b(1+2a) = -\frac{1}{2}$ so $b(1+2a) = \frac{-2a-1}{2}$ or, as $1 + 2a \neq 0,\ b = \frac{-2a-1}{2(2a+1)} = -\frac{1}{2}$.

So, as $b \in X,\ a * b \neq -\frac{1}{2}$ as required.

(ii) $*$ is associative $\Leftrightarrow a * (b * c) = (a * b) * c$ for all $a, b, c \in X$.

$$\begin{aligned} a * (b * c) &= a * (b + c + 2bc) \\ &= a + b + c + 2bc + 2(a(b + c + 2bc)) \\ &= a + b + c + 2bc + 2ab + 2ac + 4abc \end{aligned}$$

$$\begin{aligned} (a * b) * c &= (a + b + 2ab) * c \\ &= a + b + c + 2ab + 2(a + b + 2ab)c \\ &= a + b + c + 2ab + 2ac + 2bc + 4abc \end{aligned}$$

so $(a * b) * c = a * (b * c)$ for any $a, b, c \in X$

(iii) $r * s = s \Leftrightarrow r + s + 2rs = 3 \Leftrightarrow r(1 + 2s) = 0 \Leftrightarrow r = 0$ or $s = -\frac{1}{2}$. Now putting $s = -1$ we see that $r = 0$ is the only real number such that $r * s = 4 \forall s \in R$, and $0 \in X$. Furthermore $s * 0 = s$.

(iv) $a*b = 0 \Leftrightarrow a+b+2ab = 0 \Leftrightarrow b(1+2a) = a \Leftrightarrow b = \frac{a}{1+2a} \Leftrightarrow b*a = 0$

(b)

Group	order	distinguishing features
$\{e\}$	1	
Z_2	2	
Z_3	3	
Z_4	4	contains an element of order 4, and two elements of order 2
$z_2 \times Z_2$	2	3 elements of order 2
$Z)_5$	5	
Z_6	6	abelian
D_3	6	non-abelian
Z_7	7	
Z_8	8	abelian with one element of order 2
$Z_4 \times Z_2$	8	abelian with three elements of order 2
$Z_2 \times Z_2 \times Z_2$	8	abelian with seven elements of order 2
D_4	8	non-abelian with five elements of order 2
$\mathcal{Q}$	8	non-abelian with one element of order 2