Question

(a) Solve the following system of equations

$$\begin{aligned} x + 2y + 2z &= 11 \\ 2x - y + z &= 3 \\ -4x + 7y + z &= 13 \end{aligned}$$

Give a geometrical interpretation.

(b) Find <u>all</u> the 2x2 matrices A with the property that

$$A^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$$

Hence find all the solutions X of the equation

$$X^2 + 2X = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$$

where X is a 2x2 matrix.

Answer

(a) $$\left.\begin{array}{ccc|c} 1 & 2 & 2 & 11 \\ 2 & -1 & 1 & 3 \\ -4 & 7 & 1 & 13 \end{array}\right. \rightarrow \left.\begin{array}{ccc|l} 1 & 2 & 2 & 11 \\ 0 & -5 & -3 & -19 \\ 0 & 15 & 9 & 57 \leftarrow \text{This is } -3 \times R2 \end{array}\right.$$

So $$\begin{aligned} x + 2y + 2z &= 11 \\ 5y + 3z &= 19 \end{aligned}$$

Let $z = t$ then $y = \dfrac{19 - 3t}{5}$

Thus $x = 11 - 2t - 2\left(\dfrac{19 - 3t}{5}\right) = \dfrac{55 - 10t - 38 + 6t}{5} = \dfrac{17 - 4t}{5}$

So $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{17}{5} \\ \frac{19}{5} \\ 0 \end{pmatrix} + t \begin{pmatrix} \frac{-4}{5} \\ \frac{-3}{5} \\ 1 \end{pmatrix}$

This system represents three plans meeting in a common line, whose equation is the solution.

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

Thus

$$\begin{aligned} a^2 + bc &= 4 \qquad (1) \\ b(a+d) &= 0 \qquad (2) \\ c(a+d) &- 0 \qquad (3) \\ d^2 + bc &= 1 \qquad (4) \end{aligned}$$

(1) and (4) imply that $a^2 \neq d^2$ So $a + d \neq 0$. Thus $b = c = 0$ and $a = \pm 2$ and $d = \pm 1$

So there are 4 possibilities:

$$\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$$

$X^2 + 2X = \begin{pmatrix} 3 & 0 \\ 0 & 0 \end{pmatrix}$

if and only if $X^2 + 2X + I = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

Thus $(X + I)^2 = \begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$

$X + I =$ any of the 4 above.

So $X = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -2 & 0 \\ 0 & -1 \end{pmatrix}.$