Question

Find two different solutions (one can come from the general solution and the other from a singular solution) to the equation

$$\left(\frac{dy}{dx}\right)^2 - x\frac{dy}{dx} + y = 0$$

which satisfy the condition $y(1) = 1/4$. $\qquad (*)$

Answer

First rewrite in standard form. $y = x\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$, now differentiate to get

$\dfrac{dy}{dx} = \dfrac{dy}{dx} + x\dfrac{d^2y}{dx^2} - 2\dfrac{dy}{dx}\dfrac{d^2y}{dx^2} \Rightarrow 0 = \dfrac{d^2y}{dx^2}\left(x - 2\dfrac{dy}{dx}\right)$

hence

i) $\dfrac{d^2y}{dx^2} = 0$ and $\quad y = x\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$

$\Downarrow \qquad\qquad\qquad \Downarrow$

$\dfrac{dy}{dx} = c \quad \Rightarrow \quad y = cx - c^2$ $\qquad$ this is the general solution.

ii) $x - 2\dfrac{dy}{dx} = 0$ and $y = x\dfrac{dy}{dx} - \left(\dfrac{dy}{dx}\right)^2$

eliminate $\dfrac{dy}{dx} \Rightarrow y = \dfrac{x^2}{4}$ this is a singular solution.

$y(1) = \frac{1}{4} \quad \Rightarrow$ singular solution $y = \frac{x^2}{4}$is valid.

$\Rightarrow$ general solution $y = cx - c^2$with $\frac{1}{4} = A - A^2$

$\Rightarrow A = \dfrac{1}{2}$ hence $y = \dfrac{x}{2} - \dfrac{1}{4}$ is a solution.