QUESTION

Prove the following facts about the Dirichlet product $f * g$ of arithmetic functions f and g:-

(i) If f and g are multiplicative, so is $f * g$.

(ii) $f * g = g * f$.

(iii) $(f * g) * h = f * (g * h)$.

ANSWER

(i) Suppose $\gcd(m, n) = 1$. We need to prove $(f * g)(mn) = (f * g)(m)(f * g)(n)$. to see how to perform the manipulations, it helps to write both down:-

$$(f * g)(mn) = \sum_{d|mn} f(d)g\left(\frac{mn}{d}\right)$$

and

$$(f * g)(m)(f * g)(n) = \sum_{d|m} f(d)g(\left(\frac{m}{d}\right) \sum_{d|n} f(d)g\left(\frac{n}{d}\right)$$

.

It is probably easier to manipulate the second expression to get the first. First we will use different symbols for the divisors of m and the divisors of n (at present d is used for both) and then we'll combine the sums:-

$$\begin{aligned}
(f * g)(m)(f * g)(n) &= \sum_{d_1|m} f(d_1)g\left(\frac{m}{d_1}\right) \sum_{d_2|n} f(d_2)g\left(\frac{n}{d_2}\right) \\
&= \sum_{d_1|m, d_2|n} f(d_1)g\left(\frac{m}{d_1}\right) f(d_2)g\left(\frac{n}{d_2}\right) \\
&= \sum_{d_1|m, d_2|n} f(d_1)f(d_2)g\left(\frac{m}{d_1}\right) g\left(\frac{n}{d_2}\right).
\end{aligned}$$

Now $\gcd(m, n) = 1$, so if $d_1|m$ and $d_2|n$ then $\gcd(d_1, d_2) = 1$ and $\gcd\left(\frac{m}{d_1}, \frac{n}{d_2}\right) = 1$, so our sum becomes $\sum_{d_1|m, d_2|n} f(d_1 d_2)g\left(\frac{mn}{d_1 d_2}\right)$.

Now noting that as d_1 ranges over the divisors m and d_2 over the divisors of n, $d_1 d_2$ ranges over the divisors of mn (since $\gcd(m, n) = 1$), we see that this is $\sum_{d|mn} f(d)g\left(\frac{mn}{d}\right) = (f * g)(mn)$ as required.

(ii)

$$\begin{aligned}
f*g(n) &= \sum_{d|n} f(d)g\left(\frac{n}{d}\right) \\
&= \sum_{d_1d_2=n} f(d_1)g(d_2) \\
&= \sum_{d_2d_1=n} f(d_1)g(d_2) \\
&= \sum_{d_1d_2=n} f(d_2)g(d_1) \\
&= \sum_{d|n} f\left(\frac{n}{d}\right)g(d) = g*f(n).
\end{aligned}$$

(iii)

$$\begin{aligned}
((f*g)*h)(n) &= \sum_{d|n}(f*g)(d)h\left(\frac{n}{d}\right) \\
&= \sum_{d_1d_2=n} (f*g)(d_1)h(d_2) \\
&= \sum_{d_1d_2=n} \left(\sum_{d|d_1} f(d)g\left(\frac{d_1}{d}\right)\right)h(d_2) \\
&= \sum_{d_1d_2=n} \left(\sum_{de=d_1} f(d)g(e)\right)h(d_2) \\
&= \sum_{ded_d=n} f(d)g(e)h(d_2) \\
&= \sum_{d_1d_2d_3=n} f(d_1)g(d_2)h(d_3).
\end{aligned}$$

Similarly

$$\begin{aligned}
(f*(g*h))(n) &= \sum_{d|n} f(d)(g*h)\left(\frac{n}{d}\right) \\
&= \sum_{d_1d_2=n} f(d_1)(g*h)(d_2) \\
&= \sum_{d_1d_2=n} f(d_1)\left(\sum_{d|d_2} g(d)h\left(\frac{d_2}{d}\right)\right)
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{d_1 d_2 = n} f(d_1) \left(\sum_{de = d_2} g(d)h(e) \right) \\
&= \sum_{d_1 de = n} f(d_1) g(d) h(e) \\
&= \sum_{d_1 d_2 d_3 = n} f(d_1) g(d_2) h(d_3).
\end{aligned}
$$

These are the same, so $((f * g) * h) = (f * (g * h))$ as required.