QUESTION

Prove that if f and g are multiplicative, then so is the function $f.g$ defined by $f.g(n) = f(n)g(n)$ for all $n \in N$.

If $g(n) \neq 0$ for all $n \in Z$, prove that the function $\frac{f}{g}$ defined by $\frac{f}{g}(n) = \frac{f(n)}{g(n)}$ for all $n \in N$ is also multiplicative.

Deduce that the function $h(n)$ defined by $h(n) = \sum_{d|n} \frac{\mu(d)d|(d)}{\sigma(d)}$ is multiplicative. By finding the value of h when n is a prime power, find a formula for $h(n)$ in terms of the prime factorisation of n.

ANSWER

Let $\gcd(m.n) = 1$. Then $f.g(mn) = f(mn).g(mn) = f(m)f(n).g(m)g(n)$ as f and g are both multiplicative. But $f.g(m).f.g(n) = f(m)g(m)f(n)g(n)$ by definition, so comparing this with the equation above, $f.g(mn) = f.g(m)f.g(n)$, showing that $f.g$ is multiplicative, as required.

Similarly, if $\gcd(m,n) = 1$, $\frac{f}{g}(mn) = \frac{f(mn)}{g(mn)} = \frac{f(m)f(n)}{g(m)g(n)}$ by multiplicativity of f and g. But $\frac{f}{g}(m).\frac{f}{g}(n) = \frac{f(m)}{g(m)}.\frac{f(n)}{g(n)}$, and again, comparing with the equation above, $\frac{f}{g}(mn) = \frac{f}{g}(m).\frac{f}{g}(n)$, so $\frac{f}{g}$ is multiplicative.

Now μ, d and σ are all multiplicative, so by the above $\frac{\mu d}{\sigma}$ is multiplicative too, and $h(n)$ is given by $h(n) = \sum_{d|N} \left(\frac{\mu.d}{\sigma}\right)(d)$, so $h(n)$ is multiplicative by th.8.1.

Now $h(p^k) = \sum_{d|p^k} \left(\frac{\mu d}{\sigma}\right)(d) = \sum_{d|p^k} \frac{\mu(d)d(d)}{\sigma(d)}$. If $d|p^k$, then d is a power of p, so by definition of $\mu, \mu(d)$ will be 0 unless $d = 1$ or $d = p$. Thus

$$\begin{aligned} h(p^k) &= \frac{\mu(1)d(1)}{\sigma(1)} + \frac{\mu(p)d(p)}{\sigma(p)} \text{ (all other terms vanishing)} \\ &= 1 - \frac{2}{p+1} \text{ (using } \mu(1) = d(1) = \sigma(1) = 1,\ \mu(p) = -1, d(p) = 2, \\ &\quad \sigma(p) = p+1) \\ &= \frac{p-1}{p+1} \end{aligned}$$

Thus by multiplicativity $h(n) \prod_{p|n} \frac{(p-1)}{(p+1)}$.