QUESTION

Let p be a prime number and suppose that $n! = p^e s$ with $\mathrm{HCT}(p,s) = 1$ where $n!$ is the product of the integers $1, 2, \ldots, n$, as usual.

(i) Show that

$$e = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^r}\right] + \ldots$$

where $[x]$ denotes the greatest integer less than of equal to x.

(ii) If $n = a_0 + a_1 p + a_2 p^2 + \ldots + a_k p^k$, where $0 \le a_i \le p-1$ for each i, prove that e in part (i) is given by

$$e = (p-1)^{-1}(n - a_0 - a_1 - a_2 - \ldots - a_k).$$

(iii) Show that the largest power of 2 which will divide the binomial coefficient

$$\binom{2^t}{2^{t-1} - 2}$$

is 2^{t-1}, if $t \ge 3$.

ANSWER

(i) Consider the numbers $1, 2, \ldots, n$. Their product is $n!$. We can calculate e by taking each of these numbers, finding the exact power of p which divides it and then e is the sum of all these exponents. However, we are going to calculate this sum another way. Consider th following array:

$$\begin{array}{ccccc}
p, & 2p, & 3p, & \ldots & \left[\frac{n}{p}\right] p, \\
p^2, & 2p^2, & 3p^2, & \ldots & \left[\frac{n}{p^2}\right] p^2, \\
p^3, & 2p^3, & 3p^3, & \ldots & \left[\frac{n}{p^3}\right] p^3, \\
\vdots & \vdots & & & \\
p^k, & 2p^k, & 3p^k, & \ldots & \left[\frac{n}{p^k}\right] p^k \\
\vdots & \vdots & & &
\end{array}$$

The first row consists of all multiples of p less than or equal to n. The second row is all the multiples of p^2 less than or equal to n and so on.

A number which is less than or equal to n and is exactly divisible by p^s will appear exactly s times in the array- once on each of the first s rows. Hence the number of numbers in this array is exactly the sum of the exponents mentioned at the start of the proof. Clearly this number is equal to $\left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \left[\frac{n}{p^3}\right] + \ldots + \left[\frac{n}{p^r}\right] + \ldots$ as required.

(ii) We have

$$\begin{aligned}
\left[\frac{n}{p}\right] &= a_1 + a_2 p + \ldots + a_k p^{k-1}, \\
\left[\frac{n}{p^2}\right] &= a_2 + a_3 p + \ldots + a_k p^{k-2}, \\
\vdots \quad &\vdots \quad \vdots \\
\left[\frac{n}{p^{k-1}}\right] &= a_{k-1} + a_k p, \\
\left[\frac{n}{p^k}\right] &= a_k.
\end{aligned}$$

Hence

$$\begin{aligned}
e &= a_1 + a_2(1+p) + a_3(1+p+p^2) + \ldots + a_k(1+p+\ldots+p^{k-1}) \\
&= (p-1)^{-1}(a_1(p-1) + a_2(p^2-1) + \ldots + a_k(p^k-1)) \\
&= (p-1)^{-1}(n - a_0 - a_1 - a_2 - \ldots - a_k)
\end{aligned}$$

as required.

(iii) If $n \geq 3$,

$$\binom{2^n}{2^{n-1}-2} = \frac{(2^n)!}{(2^{n-1}-2)!(2^{n-1}+2)!}$$

by part (ii)

$$\begin{aligned}
(2^n)! &= (2t+1)2^{2^n-1} \\
(2^{n-1}-2)! &= (2u+1)2^{2^{n-1}-2-(n-2)} \\
(2^{n-1}+2)! &= (2v+1)2^{2^{n-1}+2-1}
\end{aligned}$$

Hence the power of 2 exactly dividing this binomial coefficient is 2^e where

$$e = 2^n - 1 - (2^{n-1} - 2 - (n-2)) - (2^{n-1} + 2 - 2) = n - 1$$