QUESTION

(i) Explaining your reasoning carefully, prove that both 311 and 317 are primes.

(ii) Stating clearly the theoretical results you have used, calculate the Legendre symbol

$$\left(\frac{317}{311}\right).$$

(Hint: You may assume that, if p is an odd prime, 2 is a square mod (p) if and only if $p \equiv \pm 1 \bmod (8)$.)

ANSWER

(i) Since $18^2 = 324, \sqrt{311}$ and $\sqrt{317}$ are both strictly smaller than 18. Hence, by Eratosthenes observation, we have only to verify that no prime among 2,3,5,7,11,13,17 divides 311 or 317. Clearly 2 does not divide any odd number. Since 3+1+1=5 and 3+1+7=11 are not divisible by 3, 3 does not divide either number. Neither ends in 0 or 5 so 5 does not divide either number. Since $7 \times 45 = 315$ w see that 7 does not divide either number. Since $3-1+1=3$ and $3-1+7=9$ are not divisible by 11, 11 does not divide either number. Finally, since $17 \times 19 = 323$, 17 does not divide either number.

(ii) The following succession of Legender symbols are equal:

$$\left(\frac{317}{311}\right) = \left(\frac{6}{311}\right)$$

since $317 \equiv 6 \bmod (311)$. Then

$$\left(\frac{6}{311}\right) = \left(\frac{2}{311}\right)\left(\frac{3}{311}\right)$$

by multiplicativity of the Legendre symbol in the top variable.

Next

$$\left(\frac{2}{p}\right) = (-1)^{(p^2-1)/8}$$

for any odd prime. When $p = 311 = (8 \times 38) + 7$ this is $(-1)^{(7^2-1)/8} = (-1)^6 = 1$. Hence we must calculate

$$\left(\frac{3}{311}\right) = \left(\frac{311}{3}\right)(-1)^{(3-1)(311-1)/4}$$

by quadratic reciprocity. However

$$\left(\frac{311}{3}\right)(-1)^{(3-1)(311-1)/4} = -\left(\frac{2}{3}\right) = -(-1) = 1.$$