QUESTION

Let a, b, c denote positive integers.

(i) Define the least common multiple, LCM(a, b), and the highest common factor, HCF(a, b), of a and b.

(ii) Prove the formula

$$LCM(a,b) = \frac{ab}{HCF(a,b)}.$$

(iii) If we define LCM(a, b, c) and HCF(a, b, c) in a similar manner which of the following two formulae is correct?

$$LCM(a,b,c) = \frac{abc}{NCF(a,b,c)}$$

or

$$LCM(a,b,c) = \frac{abcHCF(a,b,c)}{HCF(a,b)HCF(a,c)HCF(b,c)}.$$

Prove it. (Hint: In (iii) you may assume that every positive integer has a unique factorisation into prime powers.)

ANSWER

(i) LCM(a, b) is the (positive) integer, e, such that $a|e$ (a divides e) and $b|e$ and if f is any other (positive) integer multiple of both a and b then $e \leq f$.

HCF(a, b) is the (positive) integer, d, such that $d|a$ and $d|b$ and if f is any other (positive) integer dividing both a and b then $f|d$ (or, as turns out to be equivilent, $f \leq d$.)

(ii) If HCF$(a, b) = d$ there exist integers, n and m, such that $a === md$, $b = nd$. Hence $\frac{ab}{d} = mnd = mb = na$ which shows that $\frac{ab}{d}$ is a common multiple. Now suppose that f is another common multiple. Since the HCF$\left(\frac{a}{d}, \frac{b}{d}\right) = 1$ there exist integers, u and v, such that $1 = u\left(\frac{a}{d}\right) + v\left(\frac{b}{d}\right)$. Therefore

$$
\begin{aligned}
f &= fu\left(\frac{a}{d}\right) + fv\left(\frac{b}{d}\right) = fum + fvn = \left(\frac{f}{nd}\right)umnd + \left(\frac{f}{md}\right)vmnd \\
&= \left(\left(\frac{f}{b}\right)u + \left(\frac{f}{a}\right)u\right)mnd \geq mnd
\end{aligned}
$$

because $\left(\left(\frac{f}{b}\right)u + \left(\frac{f}{a}\right)v\right)$ is a n integer (necessarily positive) and so $1 \leq \left(\left(\frac{f}{b}\right)u + \left(\frac{f}{a}\right)v\right)$.

(iii) The first formula is wrong, because if we set $a = b = c = 2$ then LCM(2,2,2)=2 but $\frac{2^3}{HCF(2,2,2)} = 4$. Therefore we must prove the second formula.

Let $p_1, \ldots, p_k$ be distinct primes and write

$$
\begin{aligned}
a &= p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_k^{\alpha_k} \\
b &= p_1^{\beta_1} p_2^{\beta_2} \ldots p_k^{\beta_k} \\
c &= p_1^{\gamma_1} P_2^{\gamma_2} \ldots p_k^{\gamma_k}
\end{aligned}
$$

with the $0 \leq \alpha_i, 0 \leq \beta_i, 0 \leq \gamma_i$ for all $1 \leq i \leq k$. With this notation we have

$$
\begin{aligned}
abc &= \prod_{i=1}^{k} p_i^{\alpha_i+\beta_i+\gamma_i}, \\
HCF(a,b) &= \prod_{i=1}^{k} p_i^{min(\alpha_i,\beta_i)} \\
HCF(a,c) &= \prod_{i=1}^{k} p_i^{min(\alpha_i,\gamma_i)}, \\
HCF(b,c) &= \prod_{i=1}^{k} p_i^{min(\beta_i,\gamma_i)}, \\
HCF(a,b,c) &= \prod_{i=1}^{k} p_i^{min(\alpha_i,\beta_i,\gamma_i)}.
\end{aligned}
$$

Hence the right hand expression is equal to

$$\frac{\prod_{i=1}^{k} p_i^{\alpha_i+\beta_i+\gamma_i} p_i^{min(\alpha_i,\beta_i,\gamma_i)}}{\prod_{i=1}^{k} p_i^{min(\alpha_i,\beta_i)} p_i^{min(\alpha_i,\gamma_i)} p_i^{min(\beta_i,\gamma_i)}}.$$

If, for example, $\alpha_i \leq \beta_i \leq \gamma_i$ then

$$\alpha_i + \beta_i + \gamma_i + min(\alpha_i, \beta_i, \gamma_i) - min(\alpha_i, \beta_i) - min(\alpha_i, \gamma_i) - min(\beta_i, \gamma_i)$$

is equal to $\alpha_i + \beta_i + \gamma_i + \alpha_i - \alpha_i - \alpha_i - \beta_i = \gamma_i = max(\alpha_i, \beta)i, \gamma_i)$. Therefore the right hand expression equals

$$\prod_{i=1}^{k} p_i^{max(\alpha_i,\beta_i,\gamma_i)} = LCM(a, b, c)$$

as required.