Question

The probability density function of a random variable X is given by:

$$f(x) = \begin{cases} 0 & 0 < x < 1 \\ cx^{-2} & x \geq 1 \end{cases}$$

(a) Find the value of c;

(b) Find the distribution function $F(X)$;

(c) Find $p(X > 3)$;

(d) Find the mode of this distribution;

(e) Find the mean and standard deviation of the distribution;

(f) Find the median and interquartile range of the distribution.

Answer

$$f(x) = \begin{cases} 0 & 0 < x < 1 \\ cx^{-2} & x \geq 1 \end{cases}$$

(a) $\displaystyle\int_0^\infty f(x)\,dx = c\int_1^\infty x^{-2}\,dx = c\left[\frac{x^{-1}}{-1}\right]_1^\infty = c[1-0] = c = 1$

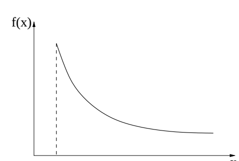
(equals one because it is a p.d.f.) Hence $c = 1$.

(b) $\displaystyle F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_1^x x^{-2}\,dx = \left[\frac{x^{-1}}{-1}\right]_1^x = 1 - \frac{1}{x}$

(c) $\displaystyle p(X > 3) = 1 - F(3) = 1 - \left(1 - \frac{1}{3}\right) = \frac{1}{3} = \int_3^\infty x^{-2}\,dx$

(d)

Mode occurs where $f(x)$ has a max.
Maximum at $x = 1$
Mode $x_{\text{mode}} = 1$



(e) Mean $\mu = \int_1^\infty x f(x)\, dx = \int_1^\infty \frac{1}{x}\, dx$

This integral diverges! The mean cannot be defined.

Standard deviation

$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \times \frac{1}{x^2}\, dx$

Similarly the standard deviation cannot be defined

[∞ will be accepted in both cases]

(f) m The median m is defined so that $F(m) = \frac{1}{2}$

$F(m) = 1 - \frac{1}{m} = \frac{1}{2} \Rightarrow m = 2$

Interquartile range between q_1 and q_3 with $F(q_1) = \frac{1}{4}$, $F(q_3) = \frac{3}{4}$

$1 - \frac{1}{q_1} = \frac{1}{4} \Rightarrow q_1 = \frac{4}{3}$

$1 - \frac{1}{q_3} = \frac{3}{4} \Rightarrow q_3 = 4$

So the interquartile range $\{q_1, q_3\} = \left\{\frac{4}{3}, 4\right\}$

Note that this is a peculiar probability density function.