Question

In each case (i), (ii) find the discriminant $\Delta \subset \mathbf{R}^2$ for the map $F : \mathbf{R}^3 \to \mathbf{R}^2$. Describe the geometric form of the set $F^{-1}(s) \subset \mathbf{R}^3$ for s belonging to each connected region of the complement of Δ in $\mathbf{R}^2$, and also for s belonging to Δ.

(i) $F(x_1, x_2, x_3) = (x_1 - (x_2^2 + x_3^2), 2x_1)$

(ii) $F(x_1, x_2, x_3) = (x_1^2 - x_2^2 - x_3^2, x_1^2 + x_2^2 + x_3^2)$.

[Hint: in (ii) let $x_2^2 + x_3^2 = r^2$.]

Answer

(i) $DF(x) = \begin{pmatrix} 1 & -2x_2 & -2x_3 \\ 2 & 0 & 0 \end{pmatrix}$ **which has rank < 2 if and only if $x_2 = x_3 = 0$, i.e singular set Σ_1 is x_1-axis.**

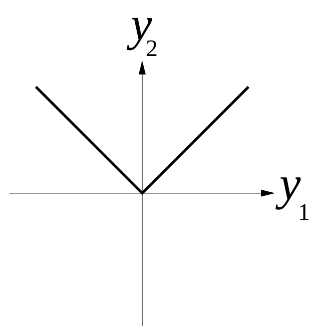
We have $F(x_1, 0, 0) = (x_1, 2x_1)$ so discriminant Δ is the line $y_2 = 2y_1$.

The set $F^{-1}(a, b)$ is the intersection of the line $x_1 = \frac{b}{2}$ with the locus $x_2^2 + x_3^2 = \frac{b}{2} - a$: cylinder if $b > 2a$, empty if $b < 2a$.

Thus $F^{-1}(a, b)$ is a circle if $b > 2a$, empty if $b < 2a$. Also one point if $b = 2a$.

(ii) $DF(x) = \begin{pmatrix} 2x_1 & -2x_2 & -2x_3 \\ 2x_1 & 2x_2 & 2x_3 \end{pmatrix}$**: rank < 2 when $x_1 = 0$ or $x_2 = x_3 = 0$.**

We have $F(0, x_2, x_3) = (x_2^2 + x_3^2)(-1, 1)$ and $F(x_1, 0, 0) = x_1^2(1, 1)$. So Δ is two half-lines.



The set $F^{-1}(a, b)$ is the intersection of the sphere $x_1^2 + x_2^2 + x_3^2 = b$ and the hyperboloid $x_1^2 - x_2^2 - x_3^2 = a$.

The 'sphere' is empty if $b < 0$, radius $\sqrt{b}$ otherwise. The hyperboloid has two sheets if $a > 0$, 1 sheet if $a < 0$ (cone when $a = 0$).

The nearest points of the hyperboloid to the origin are at distance $\sqrt{|a|}$.

Hence $F^{-1}(a, b) : \left. \begin{array}{ll} b < |a| & : \textbf{empty} \\ b > |a| & : \textbf{two circles} \end{array} \right\}$

and for $(a, b) \in \Delta$, $F^{-1}(a, b) = \left\{ \begin{array}{ll} \textbf{two points} & (\pm|a|, 0, 0) \textbf{ if } a > 0 \\ \textbf{circle} & \textbf{if } a < 0 \end{array} \right)$.