Question
Solve the diffusion equation

$$\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} = h(x,t)$$

by Laplace transforms, subject to the following boundary conditions in the specified domains:

(i) $h(x,t) = \delta(x - x_0)\delta(t - t_0),\ 0 < x < \infty,\ 0 < t < \infty$
$f(x,0) = 0,\ f(0,t) = 0,\ f,\ f_x \to 0$ as $x \to \infty$.

(ii) $h(x,t) = 0,\ -\infty < x < x\infty,\ 0 < t < \infty$
$f(x,0) = F(x),\ f(0,t) = 0,\ f,\ f_x \to 0$ as $x \to \infty$.

(iii) $h(x,t) = 0,\ 0 < x < 1,\ 0 < t < \infty$
$f(x,0) = F(x),\ f(0,t) = f(1,t) = 0$

Hints: For (i) use a Fourier sine transform in x, and a Laplace transform in t. Then follow the lecture example carefully. For (ii) Laplace transform in t not x, and then Fourier transform in x. For (iii) solve by a Laplace transform and then obtain the inverse transform either from the infinite series formula

$$\frac{1}{\sinh\sqrt{\sigma}} = 2\exp(-\sqrt{\sigma})\sum_{n=0}^{\infty}\exp(-2n\sqrt{sigma})$$

or from a series of residues (no branch cut is needed). Note these examples will be the hardest you meet, either here or on the exam!

Answer

(i) This is similar to the lecture example, only that here we have $0 < x < \infty$ rather than $-\infty < x < +\infty$. So we do a $\frac{1}{2}$ range Fourier Transform in x followed by the Laplace transform in t.

We use a <u>sine</u> transform, rather than a <u>cosine</u> transform, since $f(0,t) = 0$.

$$\begin{cases} 0 = \displaystyle\int_0^\infty \left\{\frac{\partial f}{\partial t} - \frac{\partial^2 f}{\partial x^2} - \delta(x - x_0)\delta(t - t_0)\right\}\sin kx\ dx \\ 0 = \displaystyle\int_0^\infty f(x,0)\sin kx\ dx = \hat{f}(0) \end{cases}$$

$$\Rightarrow \begin{cases} \dfrac{d\hat{f}}{dt} + k^2\hat{f} - \sin kx_0\delta(t - t_0) = 0 \\ \hat{f}(0) = 0 \end{cases}$$

Laplace Transform in t:

$$\int_0^\infty \left\{ \frac{d\hat{f}}{dt} + k^2 \hat{f} - \sin kx_0 \delta(t-t_0) \right\} = 0$$

$$\Rightarrow p\bar{\hat{f}} - \hat{f}(0) + k^2 \bar{\hat{f}} - \sin kx_0 e^{-pt_0} = 0$$

$$\underset{= 0 \text{ by b.c.}}{\uparrow}$$

$$\Rightarrow \bar{\hat{f}} = \frac{\sin kx_0 e^{-pt_0}}{(p+k^2)}$$

Invert the Laplace Transform by standard rules or inversion integral:

$$\bar{f}(k) = \frac{1}{2\pi i} \int \frac{dp \ \sin kx_0}{(p+k^2)} e^{p(t-t_0)}$$

For t > t_0 complete to left etc. · · ·
For t < t_0 complete to right etc. · · · } $u(t-t_0)$ needed

Get residue at $p = -k^2$ for $t > t_0$

$$\Rightarrow \bar{f}(k) = \sin kx_0 e^{-(t-t_0)k^2} e(t-t_0)$$

Invert the sine transform

$$f(x,t) = \frac{1}{\pi} \int_0^\infty dk \ \sin kx_0 e^{-k^2(t-t_0)} u(t-t_0) \sin kx$$

But $\sin kx_0 \sin kx = \frac{1}{2} \cos k(x-x_0) - \frac{1}{2} \cos k(x+x_0)$

So the integral $\int dk$ reduces to two integrals of the form

$$\int_0^\infty dk \ \cos kX e^{-k^2 T}; \ X = \begin{cases} x + x_0 \\ x - x_0 \end{cases}, \ T = t - t_0$$

$$= Re \left\{ \frac{1}{2} \int_{-\infty}^{+\infty} e^{-k^2 T + ikx} dk \right\} \text{ (NB good trick to know)}$$

$$= Re \left\{ \frac{1}{2} \int_{-\infty}^{+\infty} e^{-\left(kT^{\frac{1}{2}} - \frac{iX}{2T^{\frac{1}{2}}}\right)^2 - \frac{X^2}{4T}} dk \right\}$$

$$= \frac{e^{-\frac{X^2}{4T}}}{2} \sqrt{\frac{\pi}{T}}$$

$$\vdots$$

$$\Rightarrow f(x,t) = \frac{u(t-t_0)}{\sqrt{4\pi(t-t_0)}} \left\{ e^{-\frac{(x-x_0)^2}{4(t-t_0)}} - e^{-\frac{(}{x}+x_0)^2}4(t-t_0) \right\}.$$

(ii) Laplace in t and Fourier in x:

$$\begin{aligned} \int_0^\infty \frac{\partial f}{\partial t} e^{-tp}\, dt &= \int_0^\infty \frac{\partial^2 f}{\partial x^2} e^{-pt}\, dt \\ \Rightarrow p\bar{f} - f(x,0) &= \frac{\partial^2 \bar{f}}{\partial x^2} \\ p\bar{f} - F(x) &= \frac{\partial^2 \bar{f}}{\partial x^2} \end{aligned}$$

For each p this is an ODE in x for $\bar{f}(x,p)$ i.e.,

$$p\bar{f} - F(x) = \frac{d^2 \bar{f}}{dx^2}$$

Fourier transform in x (we can't solve it directly if we don't know $F(x)$)
$p\hat{\bar{f}} - \hat{F}(k) = -k^2 \hat{\bar{f}} \Rightarrow \hat{\bar{f}} = \dfrac{\hat{F}(k)}{(p+k^2)}$
Invert the Laplace transform:

$$\hat{f}(k) = \frac{1}{2\pi i} \int \frac{dp\ e^{pt} \hat{F}(k)}{p+k^2}$$

Complete to the left for $t > 0$, semicircle contour vanishes to give pole contribution from $p = -k^2$

$$\hat{f}(k) = e^{-k^2 t} \hat{F}(k).$$

Require the Fourier inversion of a product of Fourier transforms e^{-kt} and $\hat{F}(k) \Rightarrow$ convolution

$$\begin{aligned} \Rightarrow f(x,t) &= \Im^{-1}[e^{-k^2 t}] * \Im^{-1}[\hat{F}(k)] \\ f(x,t) &= \frac{1}{\sqrt{2\pi t} \int_{-\infty}^{+\infty} \underbrace{e^{-\frac{(x-y)^2}{4t}}}} F(y)\, dy \end{aligned}$$

$$\text{Fourier Transform} \left[\frac{e^{-\frac{x^2}{4t}}}{\sqrt{4\pi t}} \right] = e^{-k^2 t}$$