Question
Let $f(x, y)$ satisfy

$$\begin{cases} (x+1)\dfrac{\partial f}{\partial x} + \dfrac{\partial f}{\partial y} = x \quad \text{for x} > 0,\ \text{y} > 0 \\ f(0, y) = q(y), \quad f(x, 0) = r(x) \end{cases}$$

Show that the Laplace transform in y, $\overline{f}(x)$ say, satisfies

$$\begin{cases} (x+1)\dfrac{d\overline{f}}{dx} + p\overline{f} &= p^{-1}x + r(x) \\ \overline{f}(0) &= \overline{q} \end{cases}$$

Solve this for $\overline{f}$ in the special case

$$q(y) = y,\ r(x) = 0.$$

Use the inversion integral to calculate $f(x, y)$.

Answer
Transform the equation and boundary conditions

$$\int_0^\infty dy\ (x+1)\frac{\partial f}{\partial x}e^{-py} + \frac{[}{-}^{\infty}\frac{\partial f}{\partial y}e^{-py}dy = \int_0^\infty dy\ xe{-}py$$

becomes, by standard methods:

$$(x+1)\frac{\partial \bar{f}}{\partial x} + p\bar{f} - \underbrace{f(x, 0)}_{r(x)} = \frac{x}{p}$$

Also the boundary conditions:

$$\int_0^\infty f(0, y)e^{-py}dy = \int_0^\infty q(y)e^{-py}dy = \bar{q} = \bar{f}(0)$$

as required.
Must now solve this ODE and transformed boundary condition. ODE is linear and has an integrating factor: special case given is:

$$(x+1)\frac{\partial \bar{f}}{\partial x} + p\bar{f} = \frac{x}{p},\ \bar{f}(0) = \int_0^\infty ye^{-p}y\ dy = \frac{1}{p^2}$$

Integrating factor $= e^{\int \frac{p}{x+1}\ dx} = e^{p\ln(1+x)} = (1+x)^p$

$$
\begin{aligned}
\Rightarrow \quad \frac{\partial \bar f}{\partial x}(1+x)^p + p(1+x)^{p-1}\bar f &= \frac{x}{(1+x)p}(1+x)^p \\
\Rightarrow \quad \frac{\partial}{\partial x}[(1+x)^p \bar f] &= \frac{x}{(1+x)p}(1+x)^p \\
\Rightarrow \quad (1+x)^p \bar f &= \int \frac{x}{p}(1+x)^{p-1}\ dx \ + c \\
&= \underbrace{\frac{x(1+x)^p}{p^2} - \frac{(1+x)p+1}{p^2(p+1)}}_{\text{by integration by parts}} + c \\
\Rightarrow \quad \bar f &= \frac{x}{p^2} - \frac{(1+x)}{p^2(p+1)} + \frac{c}{(1+x)^p} \\
\bar f(0) &= \frac{1}{p^2} \\
\Rightarrow \quad \frac{1}{p^2} &= -\frac{1}{p^2(p+1)} + c \\
& \Rightarrow c = \frac{1}{p^2} + \frac{1}{p^2(p+1)}
\end{aligned}
$$

$$
\begin{aligned}
\Rightarrow \bar f &= \frac{x}{p^2} - \frac{(1+x)}{p^2(p+1)} + \frac{(p+2)}{(p+1)p^2}\frac{1}{(1+x)^p} \\
&= \frac{xp+x-1-x}{p^(p+1)} + \left[\frac{2}{p^2} - \frac{1}{p(p+1)}\right]\frac{1}{(1+x)^p} \\
&= \frac{xp+p-1-p}{p^2(p+1)} + \left[\frac{2}{p^2} - \frac{1}{p(p+1)}\right]\frac{1}{(1+x)^p} \\
&= \frac{(x+1)}{p(p+1)} - \frac{1}{p^2} + \left[\frac{2}{p^2} - \frac{1}{p(p+1)}\right]\frac{1}{(1+x)^p}
\end{aligned}
$$

So inversion integral is

$f(x,y)$

$$
= \frac{1}{2\pi i}\int dp \ \left\{\underset{(1)}{\frac{(x+1)}{p(p+1)}} - \underset{(2)}{\frac{1}{p^2}} + \underset{(3)}{\left[\frac{2}{p^2} - \frac{1}{p(p+1)}\right]\frac{1}{(1+x)^p}}\right\} e^{py}
$$

(1):

$$
\frac{1}{2\pi i}\int dp \ \frac{(x+1)}{p(p+1)} e^{py} = (x+1)[\underset{p=0}{1} \ - \ \underset{p=-1}{e^{-y}}]
$$

complete to <u>left</u> since $y > 0$. Simple poles at $p = 0,\ -1$.
Semicircular contribution $\rightarrow 0$

(2):
$\frac{1}{2\pi i}\int dp\ e^{py}{-}p^2$ complete to left since $y > 0$. Double pole at $p = 0$. Semicircular contribution $\to 0$

$= -\lim_{p\to 0}\left\{\frac{1}{1!}\frac{\partial}{\partial p}e^{py}\right\}$

$= -\lim_{p\to 0}(ye^{py})$

$= -y$

(3):
must decide whether $y = \ln(1+x) >$ or < 0. This affects which side you complete on:

$$\frac{1}{2\pi i}\int dp\ \left(\frac{2}{p^2} - \frac{1}{p(p+1)}\right)\frac{e^{py}}{(1+x)^p}$$

$$= \frac{1}{2\pi i}\int dp\left(\frac{2}{p^2} - \frac{1}{p(p+1)}\right)e^{p(y-\log(1+x))}$$

$y > \log(1+x)$: complete to left enclose poles at 0 and -1. Semicircular contribution vanishes

(3):

$$\frac{1}{2\pi i}\int dp\ \left(\frac{2}{p^2} - \frac{1}{p(p+1)}\right)e^{p(y-\log(1+x))}$$

$$= \frac{1}{2\pi i}\int dp\ \frac{2}{p^2}e^p(y-\log(1+x)) - \frac{1}{2\pi i}\int\frac{dp\ e^p(y-\log(1+x))}{p(p+1)}$$

double pole at $p = 0$ 2 simple poles at $p = 0$ and $p = -1$

$= 2(y - \log(1+x)) - 1 + e^{-y}(1+x)$

$y < \log(1+x)$: complete to the right. No poles enclosed. Semicircular contribution vanishes.

Adding (1), (2) and (3) we get:

$$f(x,y) = \begin{cases} (x+y) - 2\log(1+x) & y > \log(1+x) \\ (x+1)(1-e^{-y}) - y & y < \log(1+x) \end{cases}$$

Note that if you go back and check, $y = \log(1+x) + const$ are the characteristics of the equation. Hence you expect some funny behaviour across them. Check that $f(x,y)$ is continuous across $y = \log(1+x)$ by substituting $y = \log(1+x)$ into both expressions for $f(x,y)$ and seeing that they both reduce to

$$f(x,\ \log(1+x)) = x - \log(1+x)$$