Question
Solve the following integral equations for $\phi(x)$ by Laplace transformations.

(i) $f(x) = \int_0^x du\ (x-u)^{-\frac{1}{2}}\phi(u)$

(ii) $\phi(t) = \alpha t + \int_0^1 du\ \sin(t-u)\phi(u)$

Answer
Solve integral equations by Laplace transforming:

(i)

$$\begin{aligned} L[f] &= L\left[\underbrace{\int_0^x du\ (u-x)^{-\frac{1}{2}}\phi(u)}_{convolution}\right] \\ &\Downarrow \\ L[f(x)] &= L[x^{-\frac{1}{2}} \times L[\phi(x)] \\ \bar{f}(p) &= \frac{(-\frac{1}{2})!}{p^{\frac{1}{2}}} \times \bar{\phi}(p) \text{ from standard result} \end{aligned}$$

$\frac{(-\frac{1}{2})!}{p^{\frac{1}{2}}} = \int_0^\infty dx\ e^{-px}e^{-\frac{1}{2}}$

Thus $\bar{\phi}(p) = \frac{\sqrt{p}\bar{f}(p)}{(-\frac{1}{2})!} = \sqrt{\frac{p}{\pi}}\bar{f}(p)$

so

$$\begin{aligned} \phi(x) &= L^{-1}\left[\frac{\sqrt{p}\bar{f}(p)}{\sqrt{\pi}}\right] \\ &= L^{-1}[\bar{a}\bar{b}] \\ &= a * b \ \underline{\text{convolution}} \end{aligned}$$

where $\bar{a} = \frac{\sqrt{p}}{\sqrt{\pi}},\ \bar{b} = \bar{f}(p)$

Require $L^{-1}\left[\frac{\sqrt{p}}{\sqrt{\pi}}\right]$ and $L^{-1}[\bar{f}(p)] \xrightarrow{\text{easy}} \underline{f(x)}$

By standard result or by contour integral

$$L^{-1}\left[\frac{p^{\frac{1}{2}}}{\sqrt{\pi}}\right] = \frac{t^{-\frac{3}{2}}}{\Gamma(-\frac{1}{2})\sqrt{\pi}} = \frac{-t^{-\frac{3}{2}}}{2\sqrt{\pi}\sqrt{\pi}} = \frac{-t^{-\frac{3}{2}}}{2\pi}$$

$\Rightarrow \phi(x) = -\dfrac{1}{2\pi}\displaystyle\int_0^x d\xi\ f(x-\xi)\xi^{-\frac{3}{2}}$ is a solution of the integral equation.

(ii) Again Laplace transform:

$$\begin{aligned}
L[\phi(t)] &= L[\alpha t] = L[\sin t * \phi(t)] \\
\bar{\phi}(p) &= \frac{\alpha}{p^2} + L[\sin t] \times L[\phi(t)] \\
&\quad \text{where } \int_0^\infty \frac{\mathrm{dt}\ \mathrm{e}^{(\mathrm{p+i})\mathrm{t}}}{2\mathrm{i}} - \int_0^\infty \frac{\mathrm{dt}\ \mathrm{e}^{(\mathrm{p-i})\mathrm{t}}}{2\mathrm{i}} \\
\bar{\phi}(p) &= \frac{\alpha}{p^2} + \frac{\bar{\phi}(p)}{(1+p^2)} \\
\Rightarrow \bar{\phi}(p) &= \frac{\alpha(1+p^2)}{p^4} \\
&= \frac{\alpha}{p^4} + \frac{\alpha}{p^2}
\end{aligned}$$

so by contour integral inversion with 2 integrals which contain a fourth order pole at $p = 0$ and a second order pole at $p = 0$ respectively, or by standard methods,

$$\phi(t) = \alpha\left[\frac{t^3}{3!} + \frac{t}{1!}\right] \Rightarrow \phi(t) = \alpha t\left(1 + \frac{t^2}{6}\right)$$