Question

(i) Find the Laplace transform of the convolution integral

$$F(t) = \int_0^1 x^2 \exp(-x)\cos(t-x)dx.$$

(ii) Find the inverse Laplace transform of

$$\exp(-3p)\frac{\overline{f}(p)}{p^3}$$

Answer

(i) Use convolution result: If $f(t) = \int_0^t d\xi g(\xi)h(t-\xi)$

Laplace: $\bar{f}(p) = \bar{g}(p)\bar{h}(p)$

Thus if $F(t) = dx \ \int_0^t x^2 e^{-x}\cos(t-x)$

$L[F(t)] = L[x^2 e^{-x}] \times L[\cos x]$

$$\begin{aligned} L[x^2 e^{-x}] &= \int_0^\infty dx\ x^2 e^{-x} e^{-px} \\ &= \int_0^\infty dx\ x^2 e^{-x(p+1)} \\ &= \frac{2!}{(p+1)^3} \end{aligned}$$

from integral definition of factorial function

$$\begin{aligned} L[\cos t] &= \int_0^\infty dt\ \ \cos t e^{-pt} \\ &= \frac{1}{2}\int_0^\infty e^{-(p+i)t} + \frac{1}{2}\int_0 \infty e{-}(p+i)t \\ &= \frac{1}{2(p-i)} + \frac{1}{2(p-i)} \\ &= \frac{1}{2}\frac{(p+i+p-i)}{p^2+1} \\ &= \frac{p}{(p^2+1)} \end{aligned}$$

so $L[F(t)] = \dfrac{2}{(p+1)^3} \times \dfrac{p}{(p^2+1)} = \dfrac{2p}{(p+1)^3(p^2+1)}$

(ii)

$$\begin{aligned}
L^{-1}\left[\frac{e^{-3p}\bar{f}(p)}{p^3}\right] &= L^{-1}\left[e^{-3p}\bar{f}(p)\times\frac{1}{p^3}\right] \\
&= L^{-1}\left[L[\underbrace{u(t-3)f(t-3)}]\times L\left[\underbrace{\frac{t^2}{2}}\right]\right]
\end{aligned}$$

from $L[u(t-a)f(t-a)] = \bar{f}(p)e^{-ap}$ and $L[t^n] = \dfrac{n!}{p^{n+1}}$

So we have a L^{-1}[of product of L] $\Rightarrow$ inverse transform is a convolution:

$$\begin{aligned}
L^{-1}\left[\frac{e^{-3p}\bar{f}(p)}{p^3}\right] &= \int_0^t dx\ u(x-3)f(x-3)\frac{(x-t)^2}{2} \\
&= \frac{3}{t}dx\ f(x-3)\frac{(x-t)^2}{2} \\
\text{or} \quad &= \frac{1}{2}\int_3^t dx\ f(t-x)(x-3)^2
\end{aligned}$$