Question

For example 3 in the lectures, show that by collapsing the integration contour of the Bromwich inversion integral onto the cut, $f(t)$ can be represented as

$$f(t) = \frac{A}{\pi}\int_{-1}^{1} dy \frac{\cos(ty)}{\sqrt{1-y^2}}.$$

Answer

We have from lectures:

$$f(t) = \frac{A}{2\pi i}\underset{\uparrow}{\int} \frac{e^{pt}}{(p^2+1)^{\frac{1}{2}}}dp$$

where the contour is given by:
PICTURE

Now for $t > 0$ we can make a closed contour with a left hand semicircle, the semicircle contributing 0 in the limit as its radius $\to \infty$. This closed curve can thus be deformed as shown.
PICTURE

(Since PICTURE contributes zero the integral $\uparrow$ is unchanged by completing the contour. Hence any deformation of the closed contour= $\int$).

$\uparrow$

This leaves four pieces of curve to consider:

(i) Circle of small radius near $p = +i$. Put $p = i + \zeta = i + \epsilon e^{i\theta}$ to show that as $\epsilon \to 0$ this gives zero contribution.

(ii) Circle near $p = -i$. Do similar analysis ($p = i - \zeta = i - \epsilon e^{i\theta}$) to show it gives zero contribution.

(iii) The line to the right of the cut. As $\epsilon \to 0$ with $p = iy$ we get

$$i \int_{-1}^{+1} \frac{e^{ity}}{\sqrt{1-y^2}} dy$$

Why iy? Cut is along imaginary axis. Therefore expect discontinuity in $\arg(p)$ when g is totally imaginary. So: $\rightarrow$

(iv) On left hand side put $p = -iy$ to get

$$-i \int_{+1}^{-1} \frac{e^{-ity}}{\sqrt{1-y^2}} dy$$

So total is:

$$\begin{aligned} f(t) &= \frac{A}{2\pi i} \left\{ i \int_{-1}^{+1} \frac{e^{ity}}{\sqrt{1-y^2}} dy - i \int_{+1}^{-1} \frac{e^{-ity}}{\sqrt{1-y^2}} dy \right\} \\ &= \frac{A}{2\pi} \int_{-1}^{+1} \frac{dy}{\sqrt{1-y^2}} \left[e^{ity} + e^{-ity} \right] \\ &= \frac{A}{\pi} \int_{-1}^{+1} \frac{dy}{\sqrt{-y^2}} \cos ty \end{aligned}$$