Question

Describe briefly the behaviour of a particle moving in a simple random walk between two reflecting barriers.

Find the expected number of steps until the particle, initially at position j, reaches the upper barrier for the first time for the case $p \neq q$, p and q nonzero.

Calculate the expected number of consecutive steps for which the particle remains on the upper barrier during a visit.

Answer

Let E_j be the expected number of steps until the particle reaches the upper barrier for the first time, starting at j. (Barriers at a and $-b$)

$E_j = p(1 + E_{j+1}) + q(1 + E_{j-1}) + (1 - p - q)(1 + E_j)$

which rearranges to give

$pE_{j+1} - (p + q)E_j + qE_{j-1} = -1 \qquad -b < j < a$

The auxiliary equation is $p\lambda^2 - (p + q)\lambda + q = 0$

$$\text{i.e. } (\lambda + 1)(p\lambda - q) = 0 \Rightarrow \lambda = 1, \frac{q}{p}$$

A particular solution is $E_j = cj$ where

$pc(j + 1) - (p + q)cj + qc(j - 1) = -1$

$\Rightarrow c(p - q) = -1$ so $c = \dfrac{1}{(q - p)}$

Thus the general solution is

$$E_j = A + B\left(\frac{q}{p}\right)^j + \frac{j}{q - p}$$

The boundary conditions give

$E_a = 0 \quad$ so $\quad A + B\left(\dfrac{q}{p}\right)^a + \dfrac{a}{q - p} = 0,$

$E_{-b} = p(1 + E_{-b+1}) + (1 - p)(1 + E_{-b}) \Rightarrow pE_{-b+1} - pE_{-b} + 1 = 0$

Substituting from the general solution gives

$$B = -\left(\frac{q}{p}\right)^b \frac{q}{(q - p)^2} \qquad A = -\frac{a}{q - p} + \left(\frac{q}{p}\right)^{a+b} \frac{q}{(q - p)^2}$$

$$E_j = -\frac{a}{q - p} + \left(\frac{q}{p}\right)^{a+b} \frac{q}{(q - p)^2} - \left(\frac{q}{p}\right)^{b+j} \frac{q}{(q - p)^2} + \frac{j}{q - p}$$

Let N = number of steps on the upper barrier during a visit.

$$
\begin{aligned}
P(N = n) &= (1-q)^n \cdot q \\
E(N) &= \sum_{n=0}^{\infty} n(1-q)^n \cdot q = \frac{1-q}{q}
\end{aligned}
$$

(Summing the arithmetic-geometric series.)