Question

The following is a simple model for the spread of an epidemic through an infinite population. In a small time interval $(t, t+\delta t]$ each infected individual has, independently of other individuals, a chance $\lambda\,\delta t$ of infecting one healthy individual and a chance $1-\lambda\,\delta t$ of infecting no-one. He has a chance of recover which increases linearly with calendar time and is given by $\mu t \delta t$. λ and μ are both positive constants.

Show that the probability $p_n(t)$, that there are n infected individuals at time t, satisfies the differential-difference equation

$$p'_n(t) = (n-1)\lambda p_{n-1}(t) - n(\lambda + \mu t)p_n(t) + \mu t(n+1)p_{n+1}(t)\ \ n = 1, 2, \ldots$$

By constructing a differential equation, or otherwise, show that the expected number of infected individuals at time t, $M(t)$, is

$$M(0)e^{(\lambda t - \frac{1}{2}\mu t^2)}.$$

Describe the behaviour of $M(t)$ as t varies.

Answer

Let $p_n(t) = P(N(t) = n)$ where $N(t) =$ no. of infected patients.

$$\begin{aligned}
p_n(t+\delta t) &= P(N(t+\delta t) = n | N(t) = n-1)P(N(t) = n-1) \\
&+ P(N(t+\delta t) = n | N(t) = n)P(N(t) = n) \\
&+ P(N(t+\delta t) = n | N(t) = n+1)P(N(t) = n+1) \\
&= (n-1)\lambda\delta t(1 - \mu t(n-1)\delta t)p_{n-1}(t) \\
&\quad +(n\lambda\delta t n\mu t\delta t + (1 - n\lambda\delta t)(1 - n\mu t\delta t))p_n(t) \\
&\quad +((1 - (n+1)\lambda\delta t)\mu t(n+1)\delta t)p_{n+1}(t) + o(\delta t)
\end{aligned}$$

$$\begin{aligned}
\frac{p_n(t+\delta t) - p_n(t)}{\delta t} &= (n-1)\lambda p_{n-1}(t) - p_n(t)n(\lambda + \mu t) \\
&\quad +p_{n+1}(t)\mu t(n+1) + \frac{o(\delta t)}{\delta t}
\end{aligned}$$

Thus $p'_n(t) = (n-1)\lambda p_{n-1}(t) - n(\lambda + \mu t)p_n(t) + \mu t(n+1)p_{n+1}(t)$

$$
\begin{aligned}
M(t) &= \sum_{n=1}^{\infty} n p_n(t) \quad \text{so} \quad M'(t) = \sum_{n=1}^{\infty} n p_n'(t) \\
&= \lambda \sum_{n=1}^{\infty} n(n-1) p_{n-1}(t) - (\lambda + \mu t) \sum_{n=1}^{\infty} n^2 p_n(t) \\
&\quad + \mu t \sum_{n=1}^{\infty} n(n+1) p_{n+1}(t) \\
&= \lambda \sum_{n=1}^{\infty} (n+1) n p_n(t) - (\lambda + \mu t) \sum_{n=1}^{\infty} n^2 p_n(t) \\
&\quad + \mu t \sum_{n=1}^{\infty} (n-1) n p_n(t) \\
&= \sum_{n=1}^{\infty} p_n(t) n (\lambda - \mu t) \\
&= (\lambda - \mu t) M(t)
\end{aligned}
$$

i.e.

$$
M'(t) = (\lambda - \mu t) M(t); \quad M(t) = M(0) \exp\left(\lambda t - \frac{\mu t^2}{2}\right)
$$

$M(t)$ has a maximum of $M(0) \exp\left(\frac{\lambda^2}{2\mu}\right)$ when $t = \frac{\lambda}{\mu}$.