Question

In a single server queue, the time taken to serve a customer is exponentially distributed. New customers are discouraged by the sight of a long queue. If the queue size, including the customer being served, is n at time t, the probability of a new customer joining the queue in the time interval $(t, t+\delta t]$ is

$$\frac{\alpha}{n+1}\delta t + o(\delta t),$$

for some constant $\alpha > 0$. The probability of more than one customer joining the queue in this time interval is $o(\delta t)$.

Obtain the forward differential equations for the probability that the queue size is j after time t. Show that the equilibrium distribution of the process is a Poission distribution and find the proportion of time that the queue is empty.

Answer

If the queue size $N(t) = n \neq 0$ then the queue length changes in $(t, t+\delta t]$ by

$$\begin{array}{rll} +1 & \text{with probability} & \dfrac{\alpha}{n+1}\delta t + o(\delta t) \\ -1 & \text{with probability} & \mu\delta t + o(\delta t) \\ 0 & \text{with probability} & \left(1 - \left(\dfrac{\alpha}{n+1} + \mu\right)\delta t\right) + o(\delta t) \end{array}$$

If $N(t) = 0$ then the queue length changes by

$$\begin{array}{rll} +1 & \text{with probability} & \alpha\delta t + o(\delta t) \\ 0 & \text{with probability} & 1 - \alpha\delta t + o(\delta t) \end{array}$$

Let $p_n(t) = p(N(t) = n))$
$p_0(t + \delta t) = p_0(t)(1 - \alpha\delta t + o(\delta t)) + p_1(t)(\mu\delta t + o(\delta t))$
giving $p_0'(t) = -\alpha p_0(t) + \mu p_1(t)$

For $n = 1, 2, \ldots$

$$\begin{aligned} p_n(t+\delta t) &= p_n\left(1 - \left(\frac{\alpha}{n+1} + \mu\right)\delta t + o(\delta t)\right) \\ &+ p_{n+1}(\mu\delta t + o(\delta t)) + p_{n-1}(t)\left(\frac{\alpha}{n}\delta t + o(\delta t)\right) \\ \text{giving} \quad p_n'(t) &= -\left(\frac{\alpha}{n+1} + \mu\right)p_n(t) + \mu p_{n+1}(t) + \frac{\alpha}{n}p_{n-1}(t) \end{aligned}$$

The equilibrium distribution satisfies:

$$\begin{aligned} 0 &= -\alpha\pi_0 + \mu\pi_1 \\ 0 &= -\left(\frac{\alpha}{n+1} + \mu\right) p_n(t) + \mu\pi_{n+1} + \frac{\alpha}{n}\pi_{n-1} \end{aligned}$$

Recursive Solution of equilibrium equations:

$$\begin{aligned} 0 &= \alpha\pi_0 + \mu\pi_1 \quad \text{so} \quad \pi_1 = \frac{\alpha}{\mu}\pi_0 \\ n=1: \quad 0 &= -\left(\frac{\alpha}{2} + \mu\right)\pi_1 + \mu\pi_2 + \alpha\pi_0 \\ &= -\left(\frac{\alpha}{2} + \mu\right)\frac{\alpha}{\mu}\pi_0 + \mu\pi_2 + \alpha\pi_0 \\ \pi_2 &= \left(\frac{\alpha}{\mu}\right)^2 \frac{1}{2}\pi_0 \\ n=2: \quad 0 &= -\left(\frac{\alpha}{3} + \mu\right)\pi_2 + \mu\pi_3 + \frac{\alpha}{2}\pi_1 \\ &= -\left(\frac{\alpha}{3} + \mu\right)\left(\frac{\alpha}{\mu}\right)^2 \frac{1}{2}\pi_0 + \mu\pi_3 + \frac{\alpha^2}{2\mu}\pi_0 \\ \pi_3 &= \left(\frac{\alpha}{\mu}\right)^3 \cdot \frac{1}{3\cdot 2}\pi_0 \end{aligned}$$

Inductive step

$$\begin{aligned} 0 &= -\left(\frac{\alpha}{n+1} + \mu\right)\pi_n + \mu\pi_{n+1} + \frac{\alpha}{n}\pi_{n-1} \\ &= -\left(\frac{\alpha}{n+1} + \mu\right)\left(\frac{\alpha}{\mu}\right)^n \cdot \frac{1}{n!}\pi_0 + \mu\pi_{n+1} \\ &\quad + \frac{\alpha}{n} \cdot \left(\frac{\alpha}{\mu}\right)^{n-1} \cdot \frac{1}{(n-1)!}\pi_0 \end{aligned}$$

giving

$$\pi_{n+1} = \left(\frac{\alpha}{\mu}\right)^{n+1} \cdot \frac{1}{(n+1)!}\pi_0$$

Thus solving recursively gives

$$\pi_n = \frac{\alpha^n}{\mu^n n!}\pi_0$$

We require $\sum \pi_n = 1$ so $\pi_0 = e^{-\frac{\alpha}{\mu}}$ and $\pi_n = e^{-\frac{\alpha}{\mu}} \frac{\left(\frac{\alpha}{\mu}\right)^n}{n!}$

i.e. we have a Poisson distribution with parameter α/μ

The proportion of time the queue is empty is $\pi_0 = e^{-\frac{\alpha}{\mu}}$.