Question

A fish and chip shop has one server and room for L customers to queue, including the customer being served. The time taken to serve a customer is exponentially distributed with parameter μ, and customers arrive in a Poisson process at a rate $\frac{\mu}{2}$. A customer joins the queue if he is able to wait inside the shop, otherwise he buys his food elsewhere. Find the equilibrium distribution of the queue length and hence obtain the proportion of customers who go elsewhere.

The owner of the shop carries out improvements to the premises which enable twice as many customers to wait inside the shop. The improvements result in a doubling of the rate of arrivals of customers. What effect does this have on the proportion of customers lost?

Answer

This is an immigration - emigration process.

Let $X(t)$ = length of queue at time t, including the customer being served.

$$\begin{aligned}
P(X(t+\delta t)=n+1|X(t)=n) &= \frac{\mu}{2}\delta t+o(\delta t) \quad n=0,1,2,\ldots,L-1 \\
P(X(t+\delta t)=n-1|X(t)=n) &= \mu\,\delta t+o(\delta t) \quad n=1,2,\ldots,L \\
P(X(t+\delta t)=n|X(t)=n) &= 1-\frac{3\mu}{2}\delta t+o(\delta t) \quad n=1,2,\ldots,L-1 \\
P(X(t+\delta t)=0|X(t)=0) &= 1-\frac{\mu}{2}\delta t+o(\delta t) \\
P(X(t+\delta t)=L|X(t)=L) &= 1-\mu\,\delta t+o(\delta t)
\end{aligned}$$

The forward differential equations can be set up as in the general theory as follows:

$$\begin{aligned}
p_0'(t) &= -\frac{\mu}{2}p_0(t)+\mu p_1(t) \\
p_n'(t) &= \frac{\mu}{2}p_{n-1}(t)-\frac{3\mu}{2}p_n(t)+\mu p_{n+1}(t) \;\; n=1,2,...,L-1 \\
p_L'(t) &= \frac{\mu}{2}p_{L-1}(t)-\mu p_L(t)
\end{aligned}$$

The equilibrium distribution must satisfy:

$$\begin{aligned}
0 &= -\frac{1}{2}\pi_0+\pi_1 \\
0 &= \frac{1}{2}\pi_{n-1}-\frac{3}{2}\pi_n+\pi_{n+1} \;\; n=1,2,...,L-1 \\
0 &= \frac{1}{2}\pi_{L-1}-\pi_L
\end{aligned}$$

Solving recursively gives $\pi_k = \frac{1}{2}\pi_0 \; k = 0, ..., L$

$\sum \pi_k = 1$ gives $\pi_0 \left(\frac{1 - \left(\frac{1}{2}\right)^{L+1}}{1 - \frac{1}{2}} \right) = 1$ so $\pi_0 = \frac{\frac{1}{2}}{1 - \left(\frac{1}{2}\right)^{L+1}}$

The proportion of customers who go else where is the proportion of time the queue is full. i.e. $\pi_L = \frac{1}{(2^{L+1} - 1)}$. After improvements, the equilibrium equations are

$$\begin{aligned} 0 &= -\mu\pi_0 + \mu\pi_1 \\ 0 &= \mu\pi_{n-1} - 2\mu\pi_n + \mu\pi_{n+1} \; n = 1, ..., 2L - 1 \\ 0 &= \mu\pi_{2L-1} - \mu\pi_{2L} \end{aligned}$$

Solving recursively gives $\pi_k = \pi_0 = \frac{1}{2L+1}$. So the proportion of customers lost is

$$\pi_{2L} = \frac{1}{2L+1} > \frac{1}{2^{L+1} - 1} \; \text{for} L \geq 2$$