Question
A firm of consultants wins contracts according to a Poisson process with rate λ. The contracts yield income £Y_i $i = 1, 2, \ldots$ which are independent and identically distributed random variables with distribution

$$p\{Y = y\} = \frac{\alpha^y}{\lambda y} \quad \text{for} \;\; y = 1, 2, ..., \text{where} \;\; \alpha = 1 - e^{-\lambda}.$$

Let $X(t)$ denote the total value of the contracts obtained in a time interval of length t. Prove that the probability generating function for $X(t)$ is

$$G(z) = e^{-\lambda t}(1 - \alpha z)^{-t}, \quad \text{where} \quad -1 \leq z\alpha < 1$$

Hence, or otherwise, find the probability distribution of $X(t)$ and its mean and variance.
[Note that $\ln(1 + x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + ...$ for $-1 < x \leq 1$.]

Answer
$X(t)$ is a Compound Poisson Process.
The p.g.f. of each Y_i is

$$A(z) = \sum_{y=1}^{\infty} \frac{\alpha^y z^y}{\lambda y} = -\frac{1}{\lambda} \ln(1 - \alpha z)$$

By the theory of compound Poisson Processes $X(t)$ has p.g.f.

$$\begin{aligned} G(z) &= e^{-\lambda t} \exp(\lambda t A(z)) \\ &= e^{-\lambda t} \exp(-t \ln(1 - \alpha z)) \\ &= e^{-\lambda t}(1 - \alpha z)^{-t} \end{aligned}$$

The probability distribution of $X(t)$ is given by

$$\begin{aligned} P(X(t)) = j) &= \text{coefficient of } z^j \text{in the p.g.f.} \\ &= e^{-\lambda t} \begin{pmatrix} t + j - 1 \\ j \end{pmatrix} \alpha^j \\ &= e^{-\lambda t} \begin{pmatrix} t + j - 1 \\ j \end{pmatrix} (1 - e^{-\lambda})^j \end{aligned}$$

$E(X) = \left(\dfrac{\partial G}{\partial z}\right)_{z=1} = t(e^{\lambda} - 1) \quad (\alpha = 1 - e^{-\lambda})$
$E(X(X - 1)) = \left(\dfrac{\partial^2 G}{\partial z^2}\right)_{z=1} = t(t + 1)(e^{\lambda} - 1)^2$
$Var X = E(X(X - 1)) + E(X) - (E(X))^2 = te^{\lambda}(e^{\lambda} - 1)$