Question

(a) Give a set of axioms for a Poisson process with constant rate λ. State the distribution of the number of events, $N(t)$, in a time interval of length t units.

Show that for the overlapping time intervals $(0, t]$ and $(0, s]$ where $s < t$,

$$P(N(s) = i|N(t) = j) = \begin{pmatrix} j \\ i \end{pmatrix} \left(\frac{s}{t}\right)^i \left(1 - \frac{s}{t}\right)^{j-i},$$

where

$$\begin{pmatrix} j \\ i \end{pmatrix} = \frac{j!}{i!(j-i)!}$$

(b) An insurance company receives claims at times $t_1, t_2, t_3, \ldots$ which are points in a Poisson process with rate λ per week such that $0 < t_1 < t_2 < \ldots$. Find the probability distribution of the time until the n-th claim after the start of the financial year.

The claims at times $t_1, t_2, t_3, \ldots$ are $Y_1, Y_2, Y_3, \ldots$ respectively, which are independent random variables each having the same probability generating function $A(z)$. Show that the probability generating function for the total amount of claims which the company has to meet in the first t weeks of the financial year is

$$e^{\lambda t A(z) - \lambda t}.$$

Answer

(a) Axioms for a Poisson process.

(i) The numbers of events in disjoint time intervals are independent.

(ii) $P(1 \text{ event in } (t, t+\delta t]) = \lambda\delta t + o(\delta t)$ as $\delta t \to 0$
$P(0 \text{ events in } (t, t+\delta t]) = 1 - \lambda\delta t + o(\delta t)$ as $\delta t \to 0$

N(t) is Poisson(λt). So $P(N(t) = n) = \dfrac{(\lambda t)^n e^{-\lambda t}}{n!} \quad n = 0, 1, 2, \ldots$

$$P(N(s) = i|N(t) = j) = \frac{P(N(s) = i \;\&\; N(t) = j)}{P(N(t) = j)}$$

$$
\begin{aligned}
&= \frac{P(N(s) = i \ \& \ N(9s,t]) = j - i)}{P(N(t)) = j} \\
&= \frac{\dfrac{(\lambda s)^i e^{-\lambda s}}{i!} \times \dfrac{(\lambda(t-s))^{j-i} e^{-\lambda(t-s)}}{(j-i)!}}{\dfrac{(\lambda t)^j e^{-\lambda t}}{j!}} \\
&= \frac{j!}{i!(j-i)!} \cdot \frac{s^i (t-s)^{j-i}}{t^j} \\
&= \begin{pmatrix} j \\ i \end{pmatrix} \left(\frac{s}{t}\right)^i \left(1 - \frac{s}{t}\right)^{j-i}
\end{aligned}
$$

(b) Let T = time until n-th claim.

$$
\begin{aligned}
P(T \le t) &= 1 - P(T > t) \\
&= 1 - P(N(t) < n) \\
&= 1 - \sum_{j=0}^{n-1} p_j(t)
\end{aligned}
$$

To find the p.d.f. we need

$$
\begin{aligned}
\frac{d}{dt} P(T \le t) &= -\sum_{j=0}^{n-1} p_j'(t) \\
&= -\lambda e^{-\lambda t} - \sum_{j=1}^{n-1} \lambda e^{-\lambda t} \left(\frac{(\lambda t)^{j-1}}{(j-1)!} - \frac{(\lambda t)^j}{j!} \right) \\
&= \lambda e^{-\lambda t} \frac{(\lambda t)^{n-1}}{(n-1)!} \quad \text{i.e. } \ \Gamma(n, \lambda)
\end{aligned}
$$

The total amount of claims in the first t weeks is

$$
X(t) = Y_1 + Y_2 + .. + Y_{N(t)}
$$

Its p.g.f. is

$$
\begin{aligned}
\sum_{j=0}^{\infty} P(X(t) = j) z^j &= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} z^j P(X(t) = j | N(t) = n) P(N(t) = n) \\
&= \sum_{j=0}^{\infty} \sum_{n=0}^{\infty} z^j P(Y_1 + ... + Y_n = j) \frac{(\lambda t)^n e^{-\lambda t}}{n!} \\
&= \sum_{n=0}^{\infty} \left(\sum_{j=0}^{\infty} z^j P(Y_1 + ... + Y_n = j) \right) \frac{(\lambda t)^n e^{-\lambda t}}{n!}
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-\lambda t} \sum_{n=0}^{\infty} (A(z))^n \cdot \frac{(\lambda t)^n}{n!} \\
&= \exp(\lambda t A(z) - \lambda t)
\end{aligned}
$$