Question

The Ehrenfest model of diffusion consists of a Markov chain with states labeled 0, 1, 2, ..., d and one-step transition probabilities

$$\begin{aligned} P_{j,j-1} &= \frac{j}{d}, \qquad && j = 1, 2, \ldots, d, \\ P_{j,j+1} &= 1 - \frac{j}{d}, \qquad && j = 0, 1, \ldots, d-1. \end{aligned}$$

(i) Classify the states as periodic or aperiodic.

(ii) If initially the states 0 and 2 are likely to be occupied, each with probability $\frac{1}{2}$, find the probability distribution over the states after two steps.

(iii) Find the stationary distribution.

Answer

$$P = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & \cdots & & 0 & 0 \\ \frac{1}{d} & 0 & \frac{d-1}{d} & 0 & 0 & \cdots & & 0 & 0 \\ 0 & \frac{2}{d} & 0 & \frac{d-2}{d} & 0 & \cdots & & 0 & 0 \\ 0 & 0 & \frac{3}{d} & 0 & \frac{d-3}{d} & \cdots & & 0 & 0 \\ \vdots & & & & & & & & \\ 0 & 0 & \cdots & & & & \frac{d-1}{d} & 0 & \frac{1}{d} \\ 0 & 0 & \cdots & & & & 0 & 1 & 0 \end{pmatrix}$$

(i) All states are periodic with period 2.

(ii) Suppose $\mathbf{p}^{(0)} = \left(\frac{1}{2}, 0, \frac{1}{2}, 0, ..., 0\right)$

Then $\mathbf{p}^{(1)} = \mathbf{p}^{(0)} P = \left(0, \frac{d+2}{2d}, 0, \frac{d-2}{2d}, 0, ..., 0\right)$

$\mathbf{p}^{(2)} = \mathbf{p}^{(1)} P$

$= \left(\frac{d+2}{2d^2}, 0, \frac{(d-1)(d+2)}{2d^2} + \frac{3(d-2)}{2d^2}, 0, \frac{(d-2)(d-3)}{2d^2}, 0, ..., 0\right)$

(iii) The stationary distribution satisfies $\boldsymbol{\pi} = \boldsymbol{\pi} P$

Let $\boldsymbol{\pi} = (\pi_0, \pi_1, ..., \pi_d)$

So $\pi_0 = \frac{1}{d}\pi_1$

$$\pi_j = \frac{d-(j-1)}{d}\pi_{j-1} + \frac{j+1}{d}\pi_{j+1} \qquad j = 1, 2, .., d-1$$

$$\pi_d = \frac{1}{2}\pi_{d-1} \qquad \left(i.e.\, \pi_{j+1} = \frac{d}{j+1}\pi_j - \frac{d-(j-1)}{j+1}\pi_{j-1} \right)$$

$\pi_1 = d\pi_0,$

$\pi_2 = \frac{d}{s}\pi_1 - \frac{d}{2}\pi_0 = \frac{d(d-1)}{2}\pi_0,$

$$\pi_3 = \frac{d}{3}\pi_2 - \frac{d-1}{3}\pi_1 = \ldots = \begin{pmatrix} d \\ 3 \end{pmatrix} \pi_0, \ldots$$

$$\pi_j = \begin{pmatrix} d \\ j \end{pmatrix} \pi_0.$$

$\sum \pi_j = 1$, so $\displaystyle\sum_{j=0}^{d} \begin{pmatrix} d \\ j \end{pmatrix} \pi_0 = 1$ hence $\pi_0 = \dfrac{1}{2^d}$

Therefore

$$\pi_j = \frac{\begin{pmatrix} d \\ j \end{pmatrix}}{2^d}$$