Question
Describe briefly a Markov chain.

(a) A Markov chain $\{X_n\}$ $n = 0, 1, 2, ...$ has only two states and for $n = 1, 2, \ldots$

$$P(X_n = 1|X_{n-1} = 0) = p$$
$$P(X_n = 0|X_{n-1} = 1) = \alpha$$

Prove that

$$P(X_1 = 0|X_0 = 0, X_2 = 0) = \frac{(1-p)^2}{(1-p)^2 + p\alpha}$$

(b) A Markov chain with statespace composed of all the non-negative integers has one-step transition probabilities defined for $j = 0, 1, 2, \ldots$ by

$$P_{j,j+1} = p,$$

and

$$P_{j,0} = 1 - p,$$

where $0 < p < 1$.

Find the probability that, stating from state 0, the system will return to state 0 for the first time at the n-th step ($n \geq 1$). Hence, or otherwise, show that state 0 is positive recurrent.

Answer
A Markov chain is a sequence (X_n) of integer - valued random variables with the property that

$$P(X_n = k|X_0 = a_0, X_1 = a_1, ..., X_{n-1} = j) = P(X_n = k|X_{n-1} = j)$$

This is the Markov property.

(a) $P(X_1 = 0|X_0 = 0 \ \& \ X_2 = 0)$

$$= \frac{P(X_1 = 0 \ \& \ X_0 = 0 \ \& \ X_2 = 0)}{P(X_0 = 0 \ \& \ X_2 = 0)}$$

$$= \frac{P(X_1 = 0 \ \& \ X_0 = 0 \ \& \ X_2 = 0)}{P(X_0 = 0 \ \& \ X_1 = 0 \ \& \ X_2 = 0) + P(X_0 = 0 \ \& \ X_1 = 1 \ \& \ X_2 = 0)}$$

$$= \frac{P(X_2 = 0|X_1 = 0 \ \& \ X_0 = 0)P(X_1 = 0|X_0 = 0)P(X_0 = 0)}{\text{Num.} + P(X_2 = 0|X_1 = 1 \ \& \ X_0 = 0)P(X_1 = 1|X_0 = 0)P(X_0 = 0)}$$

[where Num. = numerator]

Using the Markov property and cancelling $P(X_0 = 0)$ this reduces to

$$\frac{(1-p)^2}{(1-p)^2 + p\alpha}$$

(b) To arrive at state 0 for the first time at step n, the Markov chain must follow the path

$$0 \to 1 \to 2 \to ... \to n-1 \to 0$$

and so the required probability is $p^{n-1}(1-p)$.

The probability of eventual return is

$$\sum_{n=1}^{\infty} p^{n-1}(1-p) = 1 \qquad (0 < p < 1)$$

The mean recurrence time is

$$(1-p)\sum_{n=1}^{\infty} np^{n-1} = \frac{1}{1-p} < \infty \text{ for } 0 < p < 1.$$

So the state 0 is positive recurrent.