Question

A household which drinks at most one bottle of wine a week keeps a stock which is checked every Saturday morning. If there is at least one bottle in stock there is probability q that a bottle will be consumed during the following week. If there is no stock then, with probability p_j, j bottles are purchased that morning ($j = 0, 1, 2, \ldots$). Sobered by the cost, the members of the household drink no wine during the following week.

The number of bottles in stock at each check forms an infinite Markov chain with states $0, 1, 2, \ldots$. Write down its probability transition matrix.

Let N_{j0} denote the number of weeks until the stock is reduced from j bottles to zero for the first time. Let $F_j(s)$ be the generating function for the probabilities:

$$f_{j0}^{(n)} = P\{N_{j0} = n\} \quad \text{for} \quad n = 1, 2, 3, ...;$$

that is

$$F_j(s) = \sum_{n=1}^{\infty} f_{j0}^{(n)} s^n.$$

Prove that

$$F_1(s) = \frac{qs}{1 - ps}, \qquad \text{where} \quad p = 1 - q.$$

By expressing N_{j0} as the sum of j independent, identically distributed random variables, or otherwise, show that

$$F_j(s) = \left[\frac{qs}{1 - ps}\right]^j \text{for} \quad j = 1, 2, 3, ...$$

Show also that

$$F_0(s) = sG\left[\frac{qs}{1 - ps}\right],$$

where $G(t)$ is the probability generating function for the distribution $\{p_j\}$ $j = 0, 1, 2, \ldots$, that is

$$G(t) = \sum_{j=0}^{\infty} p_j t^j.$$

Hence prove that return to state 0 occurs with probability 1.

Answer

The transition matrix is

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ \vdots \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & \cdots \end{array} \\ \begin{pmatrix} p_0 & p_1 & p_2 & p_3 & \cdots \\ q & 1-q & 0 & 0 & \cdots \\ 0 & q & 1-q & 0 & \cdots \\ 0 & 0 & q & 1-q & \cdots \\ \vdots & \vdots & & & \end{pmatrix} \end{array}$$

Assuming the $p_i's$ are all positive, then all states intercommunicate.
The probability that the first transition from 1 to 0 occurs in n steps is given by:

$$f_{10}^{(n)} = \begin{cases} 0 & \text{if } n = 0 \\ (1-q)^{n-1)} \cdot q & n = 1, 2, ... \end{cases}$$

The p.g.f. is therefore

$$\begin{aligned} F_1(s) &= \sum_{n=1}^{\infty} f_{10}^{(n)} s^n \\ &= \frac{q}{1-q} \sum_{n=1}^{\infty} [(1-q)s]^n \\ &= \frac{qs}{1-(1-q)s} \end{aligned}$$

Let N_{jk} denote the number of steps for the first passage from state j to state k. Now the only route from j to 0 is
$j \to (j-1) \to (j-2) \to ... \to 1 \to 0$,
with stops possible at each stage (e.g. $1 \to 1 \to 1 \to 0$)
So $N_{j0} = N_{j,j-1} + N_{j-1,j-2} + ... + N_{1,0}$
Now $N_{j,j-1}$ etc have the same distribution as $N_{1,0}$ and are independent by the Markov properly.
So the generating function for the probabilities $p_{j0}^{(n)}$ is

$$F_j(s) = [F_1(s)]^j = \left(\frac{qs}{1-ps}\right)^j \quad (p = 1-q)$$

[Alternatively

$$\begin{aligned}
f_{j,0}^{(n)} &= q f_{j-1,0}^{(n-1)} + p f_{j,0}^{(n-1)} \\
F_j(s) &= \sum_{n=1}^{\infty} f_{j,0}^{(n)} s^n \\
&= q \sum_{n=1}^{\infty} f_{j-1,0}^{(n-1)} s^n + p \sum_{n=1}^{\infty} f_{j,0}^{(n-1)} s^n \\
&= qsF_{j-1}(s) + psF_j(s) \\
\text{So } F_j(s) &= \frac{qs}{1-ps} F_{j-1}(s) \\
&= \left(\frac{qs}{1-ps}\right)^{j-1} f_1(s) \\
&= \left(\frac{qs}{1-ps}\right)^{j}
\end{aligned}$$

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Now

$$\begin{aligned}
F_0(s) &= \sum_{n=1}^{\infty} f_{00}^{(n)} s^n \\
\text{where } \ f_{(00)}^{(n)} &= \begin{cases} p_0 & n = 1 \\ \sum_{j=1}^{\infty} p_j f_{j0}^{(n-1)} & n = 2, 3, ... \end{cases} \\
\text{so } \ F_0(s) &= p_0 s + \sum_{n=2}^{\infty} \left(\sum_{j=1}^{\infty} p_j f_{j0}^{(n-1)} \right) s^n \\
&= p_0 s + \sum_{j=1}^{\infty} \sum_{n=2}^{\infty} p_j f_{j0}^{(n-1)} s^n \\
&= p_0 s + s \sum_{j=1}^{\infty} p_j \sum_{n=2}^{\infty} f_{j0}^{(n-1)} s^{(n-1)} \\
&= p_0 s + s \sum_{j=1}^{\infty} p_j \sum_{m=1}^{\infty} f_{j0}^{(m)} s^m \\
&= p_0 s + s \sum_{j=1}^{\infty} p_j F_j(s) \\
&= s \sum_{j=0}^{\infty} p_j \left(\frac{qs}{1-ps}\right)^{j}
\end{aligned}$$

$$= \quad sG\left(\frac{qs}{1-ps}\right)$$

Where G is the p.g.f. for the sequence $p_0, p_1, ...$ The probability of return to state 0 is
$F_0(1) = G\left(\frac{q}{1-p}\right) = G(1) = 1 \quad (1-p=q)$
So state 0 is recurrent.