Question

A gambler with initial capital £z plays against an opponent with capital £$(a-z)$, where a and z are integers and $0 \leq z \leq a$. At each bet the gambler wins £1 with probability p, and loses £1 with probability q or retains his stake. The bets are independent and the game ends when the gambler or his opponent is ruined. Show that the probability, P_z, that the game ends after an odd number of bets satisfies the difference equation.

$$P_z(2-p-q) + pP_{z+1} + qP_{z-1} = 1 \quad \text{for} \quad z = 1, 2, ..., a-1.$$

Two children play with a toy roulette wheel having 36 numbers $0, 1, \ldots, 35$ using stakes of 1 matchstick. One child wins if the number obtained when the wheel is spun is $0, 1, 2, \ldots, 8$ or 9 and the other child wins if the number is $10, 11, 12, \ldots, 18$ or 19. If the result is $20, 21, 22, \ldots 34$, or 35 both children retain their matchsticks. If the first child has 5 matchsticks and the second has 6, find the probability the one other of the children runs out of matchsticks after an odd number of spins, stating any assumptions made.

Answer

Arguing on the first bet conditionally gives

$$P_z = p(1-P_{z+1}) + q(1-P_{z-1}) + (1-p-q)(1-P_z)$$

i.e. $(2-p-q)P_z + pP_{z+1} + qP_{z-1} = 1$

Boundary conditions are $P_0 = 0$ and $P_a = 0$.

Let child 1 be the gambler. Then $p = \dfrac{10}{36} = \dfrac{5}{18} = q, \quad z = 5, \quad a = 11$

The equation becomes

$$5P_{z+1} + 26P_z + 5P_{z-1} = 18$$

The auxiliary equation is $5\lambda^2 + 26\lambda + 5 = 0$

$$(5\lambda + 1)(\lambda + 5) = 0 \quad \text{so} \quad \lambda = -\frac{1}{5}, -5.$$

A particular solution is $P_z = \text{constant} = \frac{1}{2}$.

So the general solution is

$$P_z = A\left(-\frac{1}{5}\right)^z + B(-5)^z + \frac{1}{2}$$

with $P_0 = 0, \quad P_{11} = 0$.

So $A + B + \frac{1}{2} = 0$,

$A\left(-\frac{1}{5}\right)^{11} + B(-5)^{11} + \frac{1}{2} = 0$

i.e. $B = \frac{5^{11}-1}{2(5^{22}-1)} \quad (\approx 1.02 \times 10^{-8})$

$A = -\frac{1}{2} - \frac{5^{11}-1}{2(5^{22}-1)}$

So

$$\begin{aligned} P_5 &= \left(-\frac{1}{2} - \frac{5^{11}-1}{2(5^{22}-1)}\right)\left(-\frac{1}{5}\right)^5 + \frac{5^{11}-1}{2(5^{22}-1)}(-5)^5 + \frac{1}{2} \\ &= 0.500128\ldots \end{aligned}$$