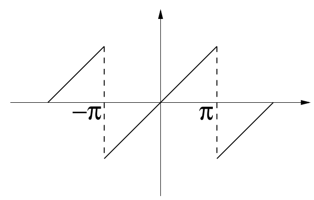
QUESTION

(b) Derive the Fourier series for the periodic function $f(t)$ which is defined by

$f(t) = t$ for $-\pi < t \leq \pi$, and $f(t + 2\pi) = f(t)$ for all t.

ANSWER

(b) Period=2π. The function is odd (from the graph) and this can be used to simplify the calculation of coefficients.



$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\,dt = 0$, since $\displaystyle\int_{-a}^{a} f(t)\,dt = 0$ when $f(t)$ is odd.

$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos(nt)\,dt = 0$, (integrand is again a odd function).

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin(nt)\,dt \\
&= \frac{2}{\pi}\int_{0}^{\pi} f(t)\sin(nt)\,dt \quad \text{(even integrand as product of odd functions)} \\
&= \frac{2}{\pi}\int_{0}^{\pi} t\sin(nt)\,dt \\
&= \frac{2}{\pi}\left\{\left[t\left(-\frac{\cos(nt)}{n}\right)\right]_0^{\pi} - \int_0^{\pi} -\frac{\cos(nt)}{n}.1\,dt\right\} \\
&= \frac{2}{\pi}\left\{-\frac{\pi}{n}\cos(n\pi) + \frac{1}{n}\left[\frac{\sin(nt)}{n}\right]_0^{\pi}\right\} \\
&= -\frac{2}{n}(-1)^n + \frac{2}{\pi n^2}(0-0) = \frac{2}{n}(-1)^{n+1}
\end{aligned}$$

Therefore $\displaystyle f(t) \sim \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n}\sin(nt)$