QUESTION

Show that the system of equations

$$\begin{aligned} x - 3y + 2z &= 5 \\ 3x + y - z &= \beta \\ \alpha x + 2y - 3z &= 15 \end{aligned}$$

has a unique solution, whatever the value of β, provided that $\alpha \neq 16$.
If $\alpha = 16$ find the value of β for which the system of equations is consistent and obtain the general solution in this case.

ANSWER

$$\begin{aligned} x - 3y + 2z &= 5 \\ 3x + y - z &= \beta \\ \alpha x + 2y - 3z &= 15 \end{aligned}$$

$$\begin{aligned} \begin{vmatrix} 1 & -3 & 2 \\ 3 & 1 & -1 \\ \alpha & 2 & -3 \end{vmatrix} &= 1(-3-(-2)) + (-3)(-\alpha - (-9)) + 2(6-\alpha) \\ &= 1(-1) - 3(9-\alpha) + 2(6-\alpha) \\ &= -1 - 27 + 3\alpha + 12 - 2\alpha \\ &= -16 + \alpha. \end{aligned}$$

$-16 + \alpha = 0$ when $\alpha = 16$. Hence there is a unique solution when $\det A \neq 0$, i.e. $\alpha \neq 16$.
When $\alpha = 16$,

$$\begin{aligned} x - 3y + 2z &= 5 \\ 3x + y - z &= \beta \\ 16x + 2y - 3z &= 1 \end{aligned}$$

$$\begin{pmatrix} 1 & -3 & 2 \\ 3 & 1 & -1 \\ 16 & 2 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ \beta \\ 15 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -7 \\ 0 & 50 & -35 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ \beta - 15 \\ -65 \end{pmatrix} \quad r_2 - 3r_1, r_3 - 16r_1$$

$$\begin{pmatrix} 1 & -3 & 2 \\ 0 & 10 & -7 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ \beta - 15 \\ 10 - 5\beta \end{pmatrix}$$

Consistent only when $10 - 5\beta = 0$, i.e. $\beta = 2$

The reduced system is

$$\begin{aligned} x - 3y + 2z &= 5 \\ 10y - 7z &= -13 \end{aligned}$$

$$z = c, \Rightarrow y = \frac{7c - 13}{10}, \Rightarrow x = 5 + 3y - 2z = 5 + \frac{21c - 39}{10} - 2c = \frac{c + 11}{10}$$

The solution is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{c+11}{10} \\ \frac{7c-13}{10} \\ c \end{pmatrix}$