QUESTION

(a) Find the general solution of the differential equation

$$(2x + te^x + \cos t)\frac{dx}{dt} + e^x - x\sin t - 2t = 0.$$

(b) Find the general solution of the second order differential equation

$$\frac{d^2x}{dt^2} + 4x = \sin(2t).$$

ANSWER

(a) $(2x + te^x + \cos t)\dfrac{dx}{dt} + e^x - x\sin t - 2t = 0$

Is this exact?

$$\left.\begin{aligned}\frac{\partial p}{\partial t} &= \frac{\partial}{\partial t}(2x + te^x + \cos t) = e^x - \sin t\\ \frac{\partial q}{\partial x} &= \frac{\partial}{\partial x}(e^x - x\sin t - 2t) = e^x - \sin t\end{aligned}\right\}\text{equal therefore exact.}$$

Find the function h to give the ODE in form $\dfrac{dh}{dt} = 0$;

$$\frac{\partial h}{\partial x} = 2x + te^x + \cos t, \qquad h = x^2 + te^x + x\cos t + f(t)$$

$$\frac{\partial h}{\partial t} = e^x - x\sin t - 2t, \qquad h = te^x + x\cos t - t^2 + g(x)$$

Thus from inspection

$$h = x^2 + te^x + x\cos t - t^2 + \text{const.}$$

The solution is $h =$const, or $x^2 + te^x + x\cos t - t^2 = \text{const.}$

(b) $\dfrac{d^2x}{dt^2} + 4x = \sin(2t)$. The auxiliary equation is $m^2 + 4 - 0,\ m = \pm 2j$.

Hence the complementary function is $x = A\cos(2t) + B\sin(2t)$.

The natural form for a particular integral is not possible (because it appears in the complementary function), so try

$$\begin{aligned} x &= t(C\cos(2t) + D\sin(2t)),\\ \frac{dx}{dt} &= C\cos(2t) + D\sin(2t) + t\{-2C\sin(2t) + 2D\cos(2t)\}\end{aligned}$$

$$
\begin{aligned}
&= (D - 2Ct)\sin(2t) + (C + 2Dt)\cos(2t), \\
\frac{d^2x}{dt^2} &= (D - 2Ct)2\cos(2t) - 2C\sin(2t) \\
&+ (C + 2Dt)(-2\sin(2t)) + 2D\cos(2t) \\
&= (4D - 4Ct)\cos(2t) + (-4C - 4Dt)\sin(2t)
\end{aligned}
$$

Substituting this into the ODE gives

$(4D - 4Ct)\cos(2t) - (4C + 4Dt)\sin(2t) + 4t(C\cos(2t) + D\sin(2t))$

$\quad = \sin(2t)$

i.e. $4D\cos(2t) - 4C\sin(2t) = \sin(2t)$.

Therefore $D = 0,\ C = -\dfrac{1}{4}$.

The general solution is $x = A\cos(2t) + B\sin(2t) - \dfrac{1}{4}t\cos(2t)$.