Question

Given $a \in \mathbf{C} - \{0\}$, set $g(z) = az$. Prove that g takes circles in $\overline{\mathbf{C}}$ to circles in $\overline{\mathbf{C}}$.

Answer

Let $\alpha z\bar{z} + \beta z + \bar{\beta}\bar{z} + \gamma = 0$ be the equation of a circle in $\mathbf{C}$, call the circle A. $(\alpha, \gamma \in \mathbf{R},\ \beta \in \mathbf{C})$.

$g(z) = az = \omega$ so $z = \dfrac{1}{a}\omega$: plug into the equation for A:

$$\alpha\left(\frac{1}{a}\omega\right)\left(\frac{1}{\bar{a}}\bar{\omega}\right) + \beta\frac{1}{a}\omega + \bar{\beta}\frac{1}{\bar{a}}\bar{\omega} + \gamma = 0$$

$$\underline{\frac{\alpha}{|a|^2}\omega\bar{\omega} + \frac{\beta}{a}\omega + \left(\frac{\bar{\beta}}{a}\right)\bar{\omega} + \gamma = 0.}$$

(note that since $\alpha = 0$ if and only if $\dfrac{\alpha}{|a|^2} = 0$ g takes circles in $\mathbf{C}$ to circles in $\mathbf{C}$ and lines in $\mathbf{C}$ to lines in $\mathbf{C}$).