Question

Using the $\delta - \epsilon$ definition of limit, prove that for each $a \in \mathbf{C} - \{0\}$,

$$\lim_{z \to \infty} az = \infty.$$

Answer

$\lim\limits_{z \to \infty} az = \infty \ (a \neq 0)$
given $\epsilon > 0$, there is $\delta > 0$ so that if $z \in \cup_\delta(\infty) - \{\infty\}$ then $az \in \cup_\epsilon(\infty)$.

$$az \in \cup_\epsilon(\infty) : \ |az| > \epsilon \ \ |z| > \frac{\epsilon}{|a|}$$

Take $\delta = \dfrac{\epsilon}{|a|}$: then if $z \in \cup_\delta(\infty) - \{\infty\}$ then $|z| > \delta$ and so $|z| > \dfrac{\epsilon}{|a|}$ and so $|az| > \epsilon$ as desired.