Question

For each point p in $\mathbf{H}$, $p \neq i$, determine the equation of the Euclidean circle or line containing the hyperbolic line through p and i, in terms of $\mathrm{Re}(p)$ and $\mathrm{Im}(p)$.

Answer

If Re(p)=0, then the hyperbolic line through p and i has the equation $\{\mathrm{Re(z)} = 0\}$. (So a vertical euclidean line.)

If Re(p)=0, the slope of the euclidean line segment through p and i is $m = \dfrac{\mathrm{Im(p)} - 1}{\mathrm{Re(p)}}$ and the midpoint is $\frac{1}{2}(p + i)$. So, the perpendicular bisector has the equation

$$y - \frac{1}{2}(\mathrm{Im(p)} + 1) = \frac{\mathrm{Re(p)}}{1 - \mathrm{Im(p)}}\left(\mathrm{x} - \frac{1}{2}\mathrm{Re(p)}\right).$$

Setting $y = 0$ and solving for x we see that the euclidean circle containing the hyperbolic line through i and p has center a

$$\begin{aligned} a &= -\frac{1}{2}(\mathrm{Im(p)} + 1)\frac{(1 - \mathrm{Im(p)})}{\mathrm{Re(p)}} + \frac{1}{2}\mathrm{Re(p)} \\ &= \frac{-1 + \mathrm{Im(p)}^2}{2\mathrm{Re(p)}} + \frac{\mathrm{Re(p)}^2}{2\mathrm{Re(p)}} = \frac{|p|^2 - 1}{2\mathrm{Re(p)}}. \end{aligned}$$

The radius of the circle is:

$$r = \left|\frac{\mathrm{Im(p)}^2}{2\mathrm{Re(p)}} - i\right| = \sqrt{\left(\frac{|p|^2 - 1}{2\mathrm{Re(p)}}\right)^2 + 1}$$