Question

For each of the following functions $f : \mathbf{R}^2 \to \mathbf{R}$ find the points (if any) where $f(x_1, x_2) = 0$ but where the 0-contour is

(i) not **locally the graph of a function $x_2 = g(x_1)$,**

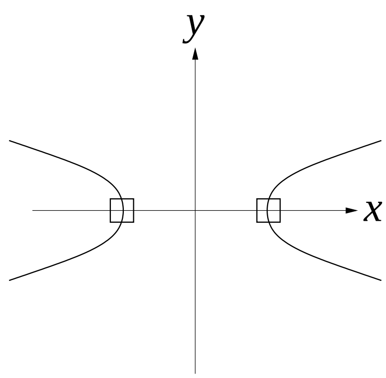
(ii) not **locally the graph of a function $x_1 = h(x_2)$,**

(iii) **both** *(i)* **and** *(ii)* **:**

$$f(x_1, x_2) = \quad \begin{array}{ll} \textbf{(a)} & x_1^2 - 2x_2^2 - 4 \\ \textbf{(b)} & x_1^3 - 3x_1 - x_2^2 + 2 \\ \textbf{(c)} & (x_1 + x_2)(x_1^2 + x_2^2 - 1) \end{array}$$

Answer

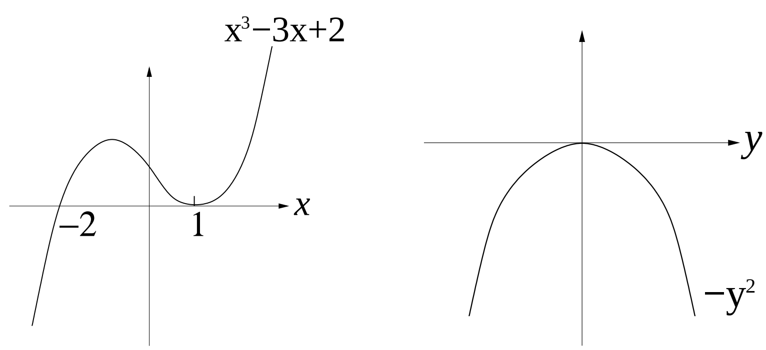
(i) $x^2 - 2y^2 = 4$

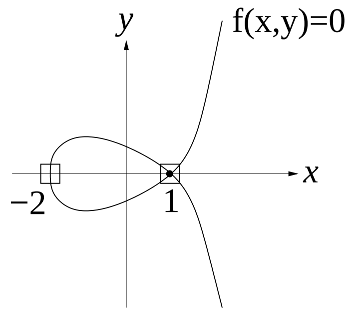


hyperbola $f(x, y) = 0$.

This can be expressed (locally) as $y = g(x)$ (smooth function g) everywhere except where the hyperbola crosses the x-axis - namely where $\frac{\partial f}{\partial y} = 0$. (There the hyperbola bends back.)

(ii)

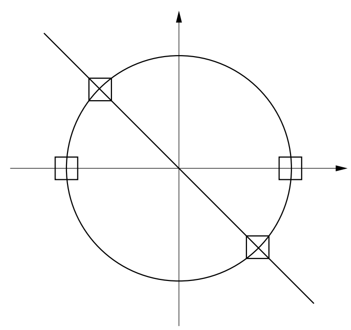




(Think of graph of f)

$\frac{\partial f}{\partial y} = -2y$ **which** $= 0$ **on the** x**-axis: points** $(-2, 0)$ **and** $(1, 0)$**. At these points** $f = 0$ **fails to have (locally) the form** $y = g(x)$**, since at** $(-2, 0)$ **it bends back, while at** $(1, 0)$ **it has two intersecting branches.**

(iii) $f(x, y) = 0$ **where** $x + y = 0$ **or** $x^2 + y^2 = 1$**.**



$\frac{\partial f}{\partial y} = 2xy + x^2 + 3y^2 - 1$**; this vanishes (on the locus** $f = 0$**) at** $(\pm 1, 0)$ **(where** $f = 0$ **bends back), and at** $\pm(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ **(where two branches intersect). Elsewhere** $y = g(x)$ **locally.**