QUESTION

A regular stream of payments (for a given number of years) is called an annuity.

(a) If the amount to be paid at the end of each year is d, and the guaranteed rate of return is r, write down the formula which gives the value of the annuity after T years.

(b) Hence deduce the size of the annuity payments necessary to accumulate a given sum in the future.

(c) Repeat the calculation for m compoundings a year.

(d) Show that the amount accumulated under continuous compounding is $\frac{d(e^{rT}-1)}{(re^{r}-1)}$.

ANSWER

(a) Pay in d at the end of each year, annual compounding for T year.

-Consider payment for year 1. It earns interest for $T-1$ years. Thus it grows to $d(1+r)^{T-1}$.

-Consider payment in year 2. It earns interest for $T-2$ years. Thus it grows to $d(1+r)^{T-2}$

$$\vdots$$

-Consider payment in year $t(<T)$. It earns interest for $T-t$ years. Thus it grows to $d(1+r)^{T-t}$.

$$\vdots$$

-Consider payment in year T. It doesn't earn interest. Thus it stays at d.

Add all these up: F_T = value at year T

$$\begin{aligned} F_T &= d(1+r)^{T-1} + d(1+r)^{T-2} + \ldots + d(1+r)^{T-t} + \ldots + d \\ &= \sum_{t=1}^{T} d(1+r)^{T-t} \end{aligned}$$

This is a geometric progression with common factor $(1+r)^{1}$ (and overall prefactor of $d(1+r)^{T}$

$$F_T = d\left[\frac{(1+r)^T - 1}{r}\right]$$

by standard sum of a GP.

(b) If you require a given sum in the future of F_T with an annual compounding at r for T years, the annual payment must be d where

$$d = \left[\frac{F_T r}{(1+r)^T - 1}\right]$$

(c) Similar arguments for m compoundings a year give

$$F_T = d\sum_{t=1}^{T}\left(1+\frac{r}{m}\right)^{m(T-t)}$$

Again this is a GP common factor $\left(1+\frac{r}{m}\right)^{-m}$

$$F_T = d\left\{\frac{\left(1+\frac{r}{m}\right)^{mT} - 1}{\left(1+\frac{r}{m}\right)^{m} - 1}\right\}$$

so

$$d = F_T\left\{\frac{\left(1+\frac{r}{m}\right)^{m} - 1}{\left(1+\frac{r}{m}\right)^{mt} - 1}\right\}$$

(d) To find the same result for continuous compounding take the limit of (c) as $m \to \infty$.

$$\lim_{m\to\infty}\left(i+\frac{r}{m}\right)^{mt} = e^{rT}$$

from question 2 above.

$$\lim_{m\to\infty}\left(1+\frac{r}{m}\right)^{m} = e^{r}\ (t=1)$$

Thus

$$F_T = d\frac{(e^{rt}-1)}{(e^{r}-1)}$$

and

$$d = \frac{F_T(e^r - 1)}{(e^{rT} - 1)}$$