QUESTION

Show that if you compound interest m times a year, with a quotes rate of f, then an amount P invested at the beginning of the year grows to $P(1+\frac{r}{m})^m$ by the end of that year. Hence show that after T years the investment has grown to $P(1+\frac{r}{m})^{mT}$. Hence show that in the limit of continuous compounding the investment grows as $P\exp(rT)$.

(Hint: $\lim_{n\to\infty}(1+\frac{1}{n})^n = e$.)

ANSWER

The annual quotes rate is r but you pay interest m times a year in installments of $\frac{r}{m}$. However you also compound m times a year. So after 1 year you have

$$P\underbrace{\left(1+\frac{r}{m}\right)\times\left(1+\frac{r}{m}\right)\times\ldots\times\left(1+\frac{r}{m}\right)}_{m\text{ times}} = P(1+\frac{r}{m})^m$$

Thus after T years you have

$$P\underbrace{\left(1+\frac{r}{m}\right)^m}_{\text{year 1}}\times\underbrace{\left(1+\frac{r}{m}\right)^m}_{\text{year 2}}\times\underbrace{\left(1+\frac{r}{m}\right)^m}_{\text{year 3}}\times\ldots\underbrace{\left(1+\frac{r}{m}\right)^m}_{\text{year } T}\times = P\left(1+\frac{r}{m}\right)^{mT}$$

Is it more than annual compounding ? Expand

$$\begin{aligned}\left(1+\frac{r}{m}\right)^{mT} &= 1+m\frac{Tr}{m}+mT(mT-1)\frac{r^2}{2m^2}+\ldots\\ &= \underbrace{1+rT}_{\text{simple}}+\frac{T^2r^2}{2}-\frac{Tr^2}{2m}\end{aligned}$$

cf. annual compounding

$$\begin{aligned}(1+r)^T &= 1+rT+T(T-1)\frac{r^2}{2}\\ &= 1+rT+\frac{T^2r^2}{2}-\frac{Tr^2}{2}\end{aligned}$$

Therefore annual compounding pays less since $m>1$.

Let $m\to\infty$. This is the limit of continuous compounding, i.e. paying interest at each and every time instant. The investment grows as

$$\lim_{m\to\infty}P\left(1+\frac{r}{m}\right)^{mT} \tag{1}$$

Now set $n = \frac{m}{r}$ with r fixed. Hence as $m \to \infty$ so does n. Thus (1) becomes

$$\begin{aligned}
\lim_{n\to\infty} P\left(1+\frac{1}{n}\right)^{nrT} &= P\left[\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n}\right]^{rt} \ (r, t \text{ fixed}) \\
&= P[e]^{rt} \text{ by the hint given} \\
&= Pe^{rt}
\end{aligned}$$

r is here called the "spot rate"
Note that the exponentials of positive numbers rT grows faster than any power of (rT) and hence this pays more than any discrete compounding which pays more than simple interest. that's why banks normally only pay annual interest, or reduced their rates for quarterly etc. payments.