Question

Determine the radius of convergence and the interval of convergence of the power series

$$\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n (x-1)^n.$$

Answer

Apply the ratio test:

$$\lim_{n\to\infty} \left| \frac{\left(1+\frac{1}{n+1}\right)^{n+1} (x-1)^{n+1}}{\left(1+\frac{1}{n}\right)^n (x-1)^n} \right| = |x-1| \lim_{n\to\infty} \frac{\left(1+\frac{1}{n+1}\right)^{n+1}}{\left(1+\frac{1}{n}\right)^n} = |x-1|\frac{e}{e} = |x-1|.$$

So, the radius of convergence is 1, and this series converges absolutely for $|x-1| < 1$. We need to check the endpoints of this interval.

At $x = 0$, the series becomes $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n (-1)^n$, which diverges by the n^{th} term test for divergence, since $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n (1)^n$ does not exist, since $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

At $x = 2$, the series becomes $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^n$, which diverges since $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$.

So, the interval of convergence is $(0, 2)$.