Question

Determine whether the infinite series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln(n)}$$

converges or diverges. (You do not need to evaluate the sum of the series in the case that it converges.)

Answer

Since the terms in the series are all positive, we may use the integral test, with $f(x) = \frac{1}{x \ln(x)}$. This function is continuous for $x \geq 3$ and is decreasing, since $f'(x) = -\frac{\ln(x)+1}{x^2 \ln^x(x)} < 0$ for $x \geq 3$. Then, the series converges if and only if the improper integral $\int_3^\infty \frac{1}{x \ln(x)} \mathrm{d}x = \lim_{M \to \infty} \int_3^M \frac{1}{x \ln(x)} \mathrm{d}x$ converges. Calculating, we see that

$$\lim_{M \to \infty} \int_3^M \frac{1}{x \ln(x)} \mathrm{d}x = \lim_{M \to \infty} \ln(\ln(x)) \Big|_3^M = \lim_{M \to \infty} (\ln(\ln(M)) - \ln(\ln(3)) = \infty.$$

Since the integral diverges, the series diverges.