Question

Determine whether the sequence

$$\left\{ a_n = \frac{\left(\frac{2}{3}\right)^n}{2 - n^{1/n}} \right\}$$

converges or diverges. If the sequence converges, determine its limit.

Answer

We know that $\lim_{n\to\infty}(\frac{2}{3})^n = 0$, since $\frac{2}{3} < 1$. Hence, we need to evaluate $\lim_{n\to\infty} n^{1/n}$: start by writing

$$n^{1/n} = \exp(\ln(n))^{1/n} = \exp\left(\frac{\ln(n)}{n}\right).$$

Since $\lim_{n\to\infty} n^{1/n} = \exp\left(\lim_{n\to\infty} \frac{\ln(n)}{n}\right)$, and since $\lim_{n\to\infty} \frac{\ln(n)}{n}$ has the indeterminate form $\frac{\infty}{\infty}$, we may use l'Hopital's rule to evaluate:

$$\lim_{n\to\infty} \frac{\ln(n)}{n} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = 0,$$

and so

$$\lim_{n\to\infty} n^{1/n} = \exp\left(\lim_{n\to\infty} \frac{\ln(n)}{n}\right) = e^0 = 1.$$

Hence, the original limit can be evaluated using the arithmetic of limits:

$$\lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n}{2 - n^{1/n}} = \frac{0}{2-1} = 0,$$

and so the sequence converges to 0.