Question
Calculate the Taylor series of the function

$$f(x) = \cos(2x)$$

about $x_0 = \pi$.

Answer
The Taylor series centered at $x_0 = \pi$ is the series

$$\sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(\pi)(x-\pi)^n.$$

Note that $f^{(n)}(\pi) = \pm \sin(\pi) = 0$ for n odd, that $f^{(4k)}(\pi) = \cos(\pi) = 1$, and that $f^{(4k+2)}(\pi) = -\cos(\pi) = -1$ for $k \geq 0$. Hence, the Taylor series becomes

$$\begin{aligned} & \sum_{k=0}^{\infty} \frac{1}{(4k)!} f^{(4k)}(\pi)(x-\pi)^{4k} + \sum_{k=0}^{\infty} \frac{1}{(4k+2)!} f^{(4k+2)}(\pi)(x-\pi)^{4k+2} \\ = & \sum_{k=0}^{\infty} \frac{1}{(4k)!}(x-\pi)^{4k} - \sum_{k=0}^{\infty} \frac{1}{(4k+2)!}(x-\pi)^{4k+2} \\ = & \sum_{p=0}^{\infty} \frac{1}{(2p)!}(-1)^p (x-\pi)^{2p}. \end{aligned}$$