Question

Consider the function $g : \mathbf{R} \to \mathbf{R}$ given by setting $g(x) = 1$ if x is a rational number and $g(x) = 0$ if x is an irrational number. Determine whether g is or is not continuous.

Answer

This function is not continuous at 0, since there are numbers arbitrarily close to 0, namely all the irrational numbers of the form $\frac{\pi}{n}$ for $n \in \mathbf{N}$, and we have that $|g(0) - g(\frac{\pi}{n})| = |1 - 0| = 1$. Hence, for $\varepsilon = \frac{1}{2}$, there does not exist $\delta > 0$ so that if $|0 - a| < \delta$, then $|g(0) - g(a)| < \varepsilon = \frac{1}{2}$. So, $\lim_{x \to 0} g(x) \neq g(0)$. (In fact, $\lim_{x \to 0} g(x)$ does not exist.)