Question
Give an example of a sequence that is bounded but not convergent, or prove that no such sequence exists. Also, give an example of a sequence that is convergent but not bounded, or prove that no such sequence exists.

Answer
The sequence $\{a_n = (-a)^n\}$ is bounded below by -1 and bounded above by 1, and so is bounded. This sequence does not converge, though; since $|a_n - a_{n+1}| = 2$ for all n, this sequence fails the Cauchy criterion, and hence diverges.

For the other part, we know that every convergent sequence is bounded. This is the proposition below. (Note that you are asked in this question to state and to write out the proof of this proposition as set out in the note below.)

Note:
Proposition
Let $A = \{a_n\}$ be a convergent sequence. Then, A is bounded.

Proof
Set $a = \lim_{n\to\infty} a_n$, and apply the definition of limit of a sequence with $\varepsilon = 1$, so that there exists $M > 0$ so that $|a_n - a| < 1$ for all $n > M$. In particular, for $n > M$, we have that a_n lies in the interval $(a-1, a+1)$. Let $s = \max(a_1, \ldots, a_M, a+1)$, and note that $a_n \leq s$ for all n. In particular, $A = \{a_n\}$ is bounded above by s.

Similarly, set $t = \min(a_1, \ldots, a_M, a-1)$, and note that $t \leq a_n$ for all n, so that $A = \{a_n\}$ is bounded below by t.

Since A is both bounded below and bounded above, it is bounded. (Note that the choice of $\varepsilon = 1$ is completely arbitrary. Any positive number will work.)