Question
Given that

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix}$$

and

$$\mathbf{c_1} = \begin{pmatrix} 1 \\ \sqrt{3} \\ 2 \\ \sqrt{3} \\ 1 \end{pmatrix}, \quad \mathbf{c_2} = \begin{pmatrix} 1 \\ -\sqrt{3} \\ 2 \\ -\sqrt{3} \\ 1 \end{pmatrix}, \quad \mathbf{c_3} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}, \quad \mathbf{c_4} = \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{c_4} = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix},$$

verify that $\mathbf{c_1}, \mathbf{c_2}, \mathbf{c_3}, \mathbf{c_4}, \mathbf{c_5}$ are all eigenvectors of A. Determine the eigenvalues corresponding to these eigenvectors.

Answer

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{3} \\ 2 \\ \sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a + \sqrt{3}b \\ 3b + \sqrt{3}a \\ 2a + 2\sqrt{3}b \\ 3b + \sqrt{3}a \\ \sqrt{3}b + a \end{pmatrix} = a + \sqrt{3}b \begin{pmatrix} 1 \\ \sqrt{3} \\ 2 \\ \sqrt{3} \\ 1 \end{pmatrix}$$

eigenvalue $= a + \sqrt{3}b$

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} 1 \\ -\sqrt{3} \\ 2 \\ -\sqrt{3} \\ 1 \end{pmatrix} = \begin{pmatrix} a - \sqrt{3}b \\ 3b - \sqrt{3}a \\ 2a - 2\sqrt{3}b \\ 3b - \sqrt{3}a \\ a - \sqrt{3}b \end{pmatrix} = a - \sqrt{3}b \begin{pmatrix} 1 \\ -\sqrt{3} \\ 2 \\ -\sqrt{3} \\ 1 \end{pmatrix}$$

eigenvalue $= a - \sqrt{3}b$

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix} = \begin{pmatrix} a + b \\ a + b \\ 0 \\ -(a + b) \\ -(a + b) \end{pmatrix} = (a + b) \begin{pmatrix} 1 \\ 1 \\ 0 \\ -1 \\ -1 \end{pmatrix}$$

eigenvalue $= a+b$

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} a-b \\ b-a \\ 0 \\ a-b \\ b-a \end{pmatrix} = (a+b) \begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

eigenvalue $= a-b$

$$\begin{pmatrix} a & b & 0 & 0 & 0 \\ b & a & b & 0 & 0 \\ 0 & b & a & b & 0 \\ 0 & 0 & b & a & b \\ 0 & 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ 0 \\ -a \\ 0 \\ a \end{pmatrix} = a \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{pmatrix}$$

eigenvalue $= a$