Question

Write the following system of simultaneous equations in matrix form and calculate the determinant as a function of a and b.

$$x + y + z = 3, \quad x + 2y + 2z = 5, \quad x + ay + bz = 3.$$

For each of the cases given below decide whether the equations have a unique solution, no solutions or infinitely many solutions; find the solutions where possible:

(i) $a = b = 1$;

(ii) $a = 1$ and $b = 1$;

(iii) $a \neq 1$ and $b = 1$;

(iv) $a = b \neq 1$;

(v) $1 \neq a \neq b \neq 1$.

Answer

In matrix form ($A\mathbf{x} = \mathbf{b}$):

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & b \end{pmatrix} \begin{pmatrix} x \\ y \\ t \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 3 \end{pmatrix}$$

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & a & b \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ -1 & 0 & 0 \\ 1 & a & b \end{vmatrix} \quad (R_2' = R_2 - 2R_1)$$

$$= (-1)(-1)^{2+1} \begin{vmatrix} 1 & 1 \\ a & b \end{vmatrix} + 0 - 0$$

(Expanding determinant along second row)

$$= b - a.$$

To solve the equations, take augmented matrix and reduce to upper triangular form using row operations.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 2 & 5 \\ 1 & a & b & 3 \end{array}\right) \xrightarrow[R_3' = R_3 - R_1]{R_2' = R_2 - R_1} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & a-1 & b-1 & 0 \end{array}\right)$$

$$\xrightarrow{R_3' = R_3 - (a-1)R_2} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & b-a & 2-2a \end{array}\right)$$

So the equations reduce to:

$$\left.\begin{array}{rcl} x+y+z &=& 3 \\ y+z &=& 2 \\ (b-a)z &=& 2-2a \end{array}\right\}$$

(i) $a = b = 1$. $\det(A) = b - a = 1 - 1 = 0$.

So either no solutions or an infinite number of solutions.

Equations become: $$\left.\begin{array}{rcl} x+y+z &=& 3 \\ y+z &=& 2 \\ 0 &=& 0 \end{array}\right\}$$

Let $z = \alpha$, so $y = 2 - \alpha$ and $x = 3 - y - z = 3 - (2 - \alpha) - \alpha = 1$.
Infinitely many solutions: $(x, y, z) = (1, 2 - \alpha, \alpha)$.

(ii) $a = 1,\ b \neq 1$. $\det(A) = b - 1 \neq 0$. So a unique solution.

Equations become: $$\left.\begin{array}{rcl} x+y+z &=& 3 \\ y+z &=& 2 \\ (b-1)z &=& 0 \end{array}\right\} \begin{array}{l} z = 0 \text{ (since b} \neq 1) \\ y = 2 \\ x = 2 \end{array}$$

Hence the unique solution: $(x, y, z) = (1, 2, 0)$.

(iii) $a \neq 1,\ b = 1$. $\det(A) = 1 - a \neq 0$, so a unique solution.

Equations become: $$\left.\begin{array}{rcl} x+y+z &=& 3 \\ y+z &=& 2 \\ (1-a)z &=& 2-2a \end{array}\right\} \begin{array}{l} z = \dfrac{2-2a}{1-a} = 2\ (1-a \neq 0) \\ y = 0 \\ x = 1 \end{array}$$

so a unique solution $(x, y, z) = (1, 0, 2)$.

(iv) $a = b \neq 1$. $\det(A) = b - a = 0$.

So either no solutions or an infinite number of solutions.

Equations become: $$\left.\begin{array}{rcl} x+y+z &=& 3 \\ y+z &=& 2 \\ 0 &=& 2-2a \end{array}\right\}$$

since $a \neq 1,\ 0 = 2 - 2a \neq 0$ which is a contradiction.

So equations are inconsistent and have no solutions.

(v) $1 \neq a \neq b \neq 1$. $\det(A) = b - a \neq 0$, so a unique solution.

Equations become: $$\left.\begin{array}{rcl} x+y+z &=& 3 \\ y+z &=& 2 \\ (b-a)z &=& 2-2a \end{array}\right\} \begin{array}{l} z = \dfrac{2-2a}{b-a}\ (b-a \neq 0) \\ y = 2 - \dfrac{2-2a}{b-a} = \dfrac{2b-2}{b-a} \\ x = 1 \end{array}$$

so a unique solution $(x, y, z) = (1, \dfrac{2b-2}{b-a}, \dfrac{2-2a}{b-a})$.